I N S T I T U T D E S T A T I S T I Q U E
B I O S T A T I S T I Q U E E T
S C I E N C E S A C T U A R I E L L E S
(I S B A)

UNIVERSITÉ CATHOLIQUE DE LOUVAIN

D I S C U S S I O N
P A P E R

2014/36

Prudence, Diversification and Optimal Portfolios

DENUIT, M. and L. EECKHOUDT

PRUDENCE, DIVERSIFICATION AND OPTIMAL PORTFOLIOS

Michel m. Denuit
Institut de statistique, biostatistique et sciences actuarielles (ISBA)
(Université Catholique de Louvain, B-1348 Louvain-la-Neuve, Belgium)
michel.denuit@uclouvain.be

Louis Eeckhoudt
IESEG School of Management (LEM, Lille, France)
and CORE (Université Catholique de Louvain, Louvain-la-Neuve, Belgium)
eeckhoudt@fucam.ac.be

September 29, 2014

Abstract

In this paper, we consider the composition of an optimal portfolio made of two dependent risky assets. The investor is assumed to be a risk-averse and prudent expected utility maximizer. We first study the conditions under which all these investors hold at least some percentage of their portfolio in one of the assets. As a byproduct we obtain conditions such that a risk-averse and prudent decision maker holds either a positive quantity of one of the assets or a proportion greater than 50% (i.e. the "50% rule"). These questions had been examined so far in the literature with no assumption beyond that of risk aversion.

Key words and phrases: Optimal asset allocation, risk aversion, downside risk, prudence.

JEL code: D81.

1 Introduction and motivation

Consider a risk-averse decision maker who has to invest a given initial wealth in two risky assets with possibly correlated performances. Negative expectation dependence introduced by Wright (1987) is a key concept to explain when all risk-averse decision makers agree to diversify their position, i.e. to invest a positive fraction of their initial wealth in each of the two assets.

The positive demand condition obtained by Wright (1987) has been extended by Hadar and Seo (1988) to ensure that the proportion of a given asset in the optimal portfolio of a risk-averse investor is at least as large as some given proportion. These conditions are closely related to the concept of marginal conditional stochastic dominance (MCSD) introduced by Yitzhaki and Olkin (1991) and Shalit and Yitzhaki (1994) as a condition under which all risk-averse investors prefer to increase the share of one risky asset over the other.

In this paper, we revisit this problem by considering risk-averse investors exhibiting prudence. In the expected utility setting, prudence is defined by the non-negativity of the third derivative of the utility function. This risk attitude was initially justified by reference to the decision of building up precautionary savings in order to better face future income risk (Kimball, 1990[1]). The role of prudence has also been illustrated in other contexts, including self-protection activities (Chiu, 2005), optimal audits (Fagart and Sinclair-Desgagné, 2007), or decreasing sensitivity to an increase in correlation when the initial wealth increases (Denuit and Rey, 2010). Quite surprisingly, although prudence originates in savings problems, this concept does not seem to have been applied to optimal portfolio selection so far. The present paper investigates the role of this additional assumption of prudence in asset allocation.

By restricting the class of risk-averse decision makers to the subset of those investors exhibiting prudence, we can weaken the condition imposed by Hadar and Seo (1988) on the joint probability distribution of the assets ensuring that the optimal portfolio comprises at least a given percentage of one of them. This weakening of the positive demand condition in Wright (1987) or of the MCSD criterion also produces new interpretations of the results. As is well known, the results obtained both by Wright (1987) and with the MCSD criterion for risk-averse decision makers can be interpreted in terms of covariances between asset returns and payoffs of digital options written on the performances of the reference portfolio comprising the desired proportion of the assets when the expected returns of the risky assets are equal. When one assumes prudence beyond risk aversion, digital options are replaced by European put options written on the reference portfolio. As such puts can theoretically be replicated by means of digital options protecting against weak performances of the reference portfolio, we thus get a weaker condition.

As is now well known, prudence is one of the risk attitudes beyond risk aversion and this is related to the notion of higher-order risk apportionment, as defined by Eeckhoudt and Schlesinger (2006). The notion of risk apportionment is a preference for a particular class of lotteries combining sure reductions in wealth and zero-mean risks. These higher-order risk attitudes entail a preference for combining relatively good outcomes with bad ones and can be interpreted as a desire to disaggregate the harms of unavoidable risks and losses. Risk

[1]Kimball (1990) coined the term prudence in his analysis of savings under future income risk. However, as indicated by Kimball (1990), this question had already been analyzed earlier e.g. by Drèze and Modigliani (1972), Sandmo (1970) and Leiland (1968).

apportionments of orders 2 and 3 correspond to risk aversion and prudence, respectively. Increasing the order of risk apportionment further restricts the class of investors and thus gives weaker conditions on the joint probability distribution of the assets.

The remainder of this paper is organized as follows. Section 2 describes the problem investigated in the present paper. Section 3 gives the solution for risk-averse decision makers summarizing the results available in the literature. These results are derived here in a unified way, allowing for an extension in Section 4 to risk-averse investors exhibiting prudence. The final Section 5 discusses the results obtained in this paper. Some technical results are gathered in the appendix to this paper. Appendix A contains the derivation of a new expansion formula which appears to be critical in obtaining the results presented in this paper. Also, an extension of the results obtained for risk-averse and prudent decision makers to investors exhibiting risk apportionment of orders 1 to $4, 5, \ldots$ is proposed in Appendix B.

2 Optimal asset allocation

Consider the following standard 2-asset portfolio problem as described e.g. in Hadar and Seo (1988). Let X_j, $j = 1, 2$, be the random return per monetary unit invested in risky asset j valued in some interval $[a, b]$ of the real line. Assume that the initial wealth is equal to unity and must be invested in one of these two risky assets by a non-satiated risk-averse decision maker. This agent is assumed to act in order to maximize the expected utility of terminal wealth which is the end-of-period value $\lambda X_1 + (1 - \lambda)X_2$ of the portfolio, where λ represents the fraction of the initial wealth invested in asset 1.

Define $\mathcal{U}_{\text{ra}}$ as the class of all utility functions $u : [a, b] \to \mathbb{R}$ with first derivative $u' \geq 0$ and second derivative $u'' \leq 0$, expressing risk aversion. For an investor with utility function $u \in \mathcal{U}_{\text{ra}}$, the optimal λ maximizes the objective function

$$\mathcal{O}(\lambda) = E[u(\lambda X_1 + (1 - \lambda)X_2)].$$

The first-order condition is

$$\mathcal{O}'(\lambda) = 0 \Leftrightarrow E[(X_1 - X_2)u'(\lambda X_1 + (1 - \lambda)X_2)] = 0. \tag{2.1}$$

Denote as $\lambda^\star$ the solution to equation (2.1), assumed to be unique. Notice that the concavity of u ensures that the objective function is also concave. The concavity of $\mathcal{O}(\lambda)$ plays a central role in the developments appearing in the remainder of this paper.

3 Risk-averse investors

The next Proposition 3.1 summarizes the results obtained so far in the literature in the case of risk-averse decision makers. Precisely, it provides different conditions ensuring that the optimal share $\lambda^\star$ invested in the first asset by every risk-averse decision maker is at least equal to some given percentage π. For the sake of completeness and to ease the extension to prudent investors, we provide a direct proof of these results, based on a useful expansion formula given in Appendix A. Henceforth, we denote as $I[\cdot]$ the indicator function, i.e. $I[A] = 1$ if event A is realized and $I[A] = 0$ otherwise.

Proposition 3.1. *Consider a fixed percentage* $\pi \in [0,1]$. *Define the reference portfolio* $\overline{X}_\pi = \pi X_1 + (1-\pi)X_2$ *comprising asset 1 in proportion* π. *The optimal share* $\lambda^\star$ *invested in the first asset is at least equal to* π *for every* $u \in \mathcal{U}_{ra}$ *if, and only if, one of the following equivalent conditions is fulfilled:*

$$\begin{aligned} & E\big[X_1 I[\overline{X}_\pi \leq z]\big] \geq E\big[X_2 I[\overline{X}_\pi \leq z]\big] \text{ for all } z \in [a,b] & (3.1)\\ \Leftrightarrow\ & Cov\big[X_1 - X_2, I[\overline{X}_\pi \leq z]\big] \geq E[X_2 - X_1]P[\overline{X}_\pi \leq z] \text{ for all } z \in [a,b]. & (3.2)\end{aligned}$$

Proof. Considering (2.1), the concavity of the objective function $\mathcal{O}$ ensures that $\lambda^\star \geq \pi$ when

$$\mathcal{O}'(\pi) \ = \ E[(X_1 - X_2)u'(\overline{X}_\pi)] \geq 0. \tag{3.3}$$

Let us apply formula (A.1) in appendix to $Z_1 = X_1$, $Z_2 = \overline{X}_\pi$ and $g(z_1,z_2) = z_1 u'(z_2)$. This gives

$$\begin{aligned} E[X_1 u'(\overline{X}_\pi)] \ &= \ u'(b)E[X_1] - \int_a^b bu''(z_2)P[\overline{X}_\pi \leq z_2]dz_2 \\ &\quad + \int_a^b u''(z_2)E\big[(b - X_1)I[\overline{X}_\pi \leq z_2]\big]dz_2. \end{aligned} \tag{3.4}$$

Hence,

$$E[(X_1 - X_2)u'(\overline{X}_\pi)] = u'(b)E[X_1 - X_2] - \int_a^b u''(z_2)E\big[(X_1 - X_2)I[\overline{X}_\pi \leq z_2]\big]dz_2.$$

As a consequence, if (3.1) is valid then condition (3.3) is fulfilled for every $u \in \mathcal{U}_{\text{ra}}$. Conversely, if (3.3) holds for all $u \in \mathcal{U}_{\text{ra}}$ then it holds in particular for the utility function $u(x) = \min\{x, z\}$ such that $u'(x) = I[x \leq z]$, which shows that inequality (3.1) must hold true. To get the equivalence (3.2), it suffices to notice that

$$\begin{aligned} E\big[(X_1 - X_2)I[\overline{X}_\pi \leq z]\big] \ &= \ E[X_1 - X_2]E\big[I[\overline{X}_\pi \leq z]\big] + Cov\big[X_1 - X_2, I[\overline{X}_\pi \leq z]\big] \\ &= \ E[X_1 - X_2]P[\overline{X}_\pi \leq z] + Cov\big[X_1 - X_2, I[\overline{X}_\pi \leq z]\big]. \end{aligned}$$

This ends the proof. □

Notice that (3.1) ensures that $E[X_1] \geq E[X_2]$ holds, by letting z tend to b. The condition $E[X_1] \geq E[X_2]$ rules out the cases where X_2 dominates X_1 by first-order stochastic dominance. Second-order stochastic dominance is nevertheless possible provided $E[X_1] = E[X_2]$.

Condition (3.1) can be found in Theorem 3 by Hadar and Seo (1988). Instead of the expansion used in the proof provided here, Hadar and Seo (1988) reduce in their Theorem 1 $\mathcal{U}_{\text{ra}}$ to the subset of all representative risk averters whose utility functions consist of two linear pieces (i.e. of the form $\min\{x, z\}$ for some fixed z). The alternative proof provided here appears useful when extending the analysis to prudent investors in Section 4.

Given a portfolio $\overline{X}_\pi$, Shalit and Yitzhaki (1994) have established that it is optimal for every risk-averse decision maker to increase the weight of asset 1 at the expense of asset 2 if, and only if,

$$E[X_1 | \overline{X}_\pi \leq z] \geq E[X_2 | \overline{X}_\pi \leq z] \text{ for all } z \tag{3.5}$$

which is obviously equivalent to condition (3.1). This condition is known in the literature as marginal conditional stochastic dominance (MCSD). In words, MCSD favors assets performing on average better in adverse situations (i.e. when the portfolio underperforms $\Leftrightarrow \overline{X}_\pi \leq z$).

If $E[X_1] = E[X_2]$ then only the covariance remains in (3.2) which reduces to

$$\lambda^\star \geq \pi \Leftrightarrow Cov\big[X_1, I[\overline{X}_\pi \leq z]\big] \geq Cov\big[X_2, I[\overline{X}_\pi \leq z]\big] \text{ for all } z.$$

This condition can be interpreted as follows. The indicator $I[\overline{X}_\pi \leq z]$ is the payoff of a digital option paying 1 if the performance of the portfolio $\overline{X}_\pi$ does not reach the threshold z. This digital option protects the investor against weak performances of the portfolio $\overline{X}_\pi$. Now, the optimal proportion invested in asset 1 is larger than π if the covariance between X_1 and the payoff of this digital option is always larger than the corresponding covariance with X_2 whatever the performance threshold z. If X_1 and X_2 are identically distributed, or simply have the same variance, then the dominating asset in the portfolio is the one which is less correlated with the payoff of the digital option written on the performance of the reference portfolio $\overline{X}_\pi$.

Example 3.2 (Positive demand)**.** The particular case $\pi = 0$ has been considered by Wright (1987) who established that all risk-averse investors hold a positive amount of each asset in their expected utility maximizing portfolio when (3.1)-(3.2) hold with $\pi = 0$, so that the reference portfolio $\overline{X}_\pi$ reduces to X_2.

Example 3.3 (The 50% rule)**.** Let us now consider the 50% rule corresponding to $\pi = 0.5$. In this case, we denote $\overline{X}_{0.5}$ simply as $\overline{X} = \frac{X_1+X_2}{2}$. Portfolio $\overline{X}$ is the equally-weighted portfolio comprising an equal share of both assets. Condition (3.1) ensuring that $\lambda^\star \geq \frac{1}{2}$ is equivalent to condition (9) in Clark and Jokung (1999), which in turn is equivalent to the MCSD criterion (3.5).

Example 3.4. If X_1 and X_2 are mutually independent then there is a positive demand for X_1 as

$$E[X_1|X_2 \leq z] = E[X_1] \geq E[X_2] \geq E[X_2|X_2 \leq z] \text{ for all } z$$

so that condition (3.1) with $\pi = 0$ (hence $\overline{X}_\pi = X_2$) is fulfilled.

Remark 3.5. In his conclusion, Wright (1987) suggested to define X_1 as more negatively expectation dependent than X_2 on $\overline{X}_\pi$ when the inequality

$$E[X_1|\overline{X}_\pi \leq t] - E[X_1] \geq E[X_2|\overline{X}_\pi \leq t] - E[X_2] \tag{3.6}$$

holds for all t. If $E[X_1] = E[X_2]$ and (3.1) holds then (3.6) is necessary fulfilled. Condition (3.6) allows one to derive inequalities involving covariances, as shown next. Consider a decreasing transformation g. From the proof of Theorem 3.1 in Wright (1987), we can write

$$\begin{aligned} Cov[X_1, g(\overline{X}_\pi)] &= \int_a^b \Big(E[X_1] - E[X_1|\overline{X}_\pi \leq x_2]\Big) P[\overline{X}_\pi \leq x_2] g'(x_2) dx_2 \qquad (3.7) \\ &\geq \int_a^b \Big(E[X_2] - E[X_2|\overline{X}_\pi \leq x_2]\Big) P[\overline{X}_\pi \leq x_2] g'(x_2) dx_2 \text{ under (3.6)} \\ &= Cov[X_2, g(\overline{X}_\pi)]. \end{aligned}$$

Hence, (3.6) ensures that the inequality $Cov[X_1, g(\overline{X}_\pi)] \geq Cov[X_2, g(\overline{X}_\pi)]$ is valid for every decreasing transformation g. The reverse inequality holds for an increasing g. Condition (3.6) corresponds to Definition 2.9 in Dionne, Li and Okou (2012), restricted to the pairs of random variables $(X_1, \overline{X}_\pi)$ and $(X_2, \overline{X}_\pi)$. As the law of total probability gives

$$E[X_1] = E[X_1|\overline{X}_\pi \leq t]P[\overline{X}_\pi \leq t] + E[X_1|\overline{X}_\pi > t]P[\overline{X}_\pi > t],$$

we have

$$P[\overline{X}_\pi \leq t]\big(E[X_1] - E[X_1|\overline{X}_\pi \leq t]\big) = P[\overline{X}_\pi > t]\big(E[X_1|\overline{X}_\pi > t] - E[X_1]\big)$$

so that condition (3.6) above can be equivalently rewritten as

$$E[X_1|\overline{X}_\pi > t] - E[X_1] \leq E[X_2|\overline{X}_\pi > t] - E[X_2].$$

Remark 3.6. It is easily seen that the optimal share $\lambda^\star$ invested in the first asset is at most equal to π if $\mathcal{O}'(\pi) \leq 0$, so that Proposition 3.1 can easily be rephrased in terms of maximum demand.

4 Risk-averse and prudent investors

Thanks to the expansion formula proposed in Appendix A, we have been able to review in a unified way the results obtained from the end of the 1980s for the optimal allocation of a risky portfolio when decision makers are risk-averse. In the present section, we now make the additional assumption that decision makers are prudent (i.e. the third derivative u''' of u is such that $u''' \geq 0$) and we examine the implication of this additional and now well-accepted assumption for optimal portfolio selection.

To motivate the analysis, we start with a simple numerical example. Consider for instance assets with respective returns given by

$$X_1 = \begin{cases} 1.1 \text{ with probability } \frac{3}{4} \\ 1.3 \text{ with probability } \frac{1}{4} \end{cases} \text{ and } X_2 = \begin{cases} 1 \text{ with probability } \frac{1}{4} \\ 1.2 \text{ with probability } \frac{3}{4} \end{cases}.$$

Both returns have the same mean and variance. Return X_1 is preferred by all risk-averse and prudent decision makers. Assume that the returns X_1 and X_2 are correlated, with joint distribution

$$\begin{aligned} P[X_1 = 1.1, X_2 = 1] &= \frac{3}{16} + \rho \\ P[X_1 = 1.1, X_2 = 1.2] &= \frac{9}{16} - \rho \\ P[X_1 = 1.3, X_2 = 1] &= \frac{1}{16} - \rho \\ P[X_1 = 1.3, X_2 = 1.2] &= \frac{3}{16} + \rho, \end{aligned} \tag{4.1}$$

for some correlation parameter $\rho \in \left[-\frac{3}{16}, \frac{1}{16}\right]$. The strength of the dependence between X_1 and X_2 is controlled by the parameter ρ. A positive (resp. negative) ρ entails positive

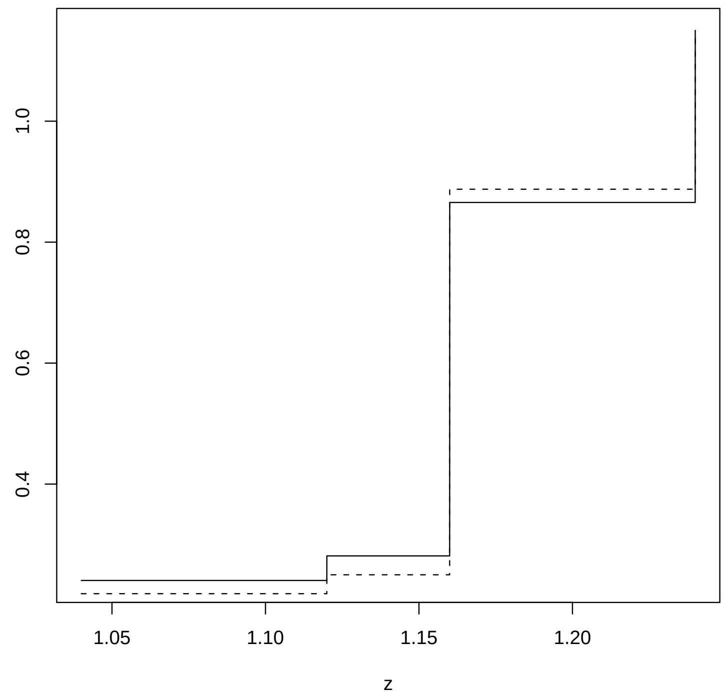

Figure 4.1: Graphs of $z \mapsto E\big[X_1 I[\overline{X}_\pi \leq z]\big]$ (continuous line —) and $z \mapsto E\big[X_2 I[\overline{X}_\pi \leq z]\big]$ (broken line - - -) for $\pi = 0.4$ when the joint distribution of asset returns X_1 and X_2 is given by (4.1) with $\rho = \frac{1}{32}$.

(resp. negative) dependence between X_1 and X_2, i.e. a large return for asset 1 tends to be accompanied by a large (resp. small) return for asset 2. The special case $\rho = 0$ corresponds to independence.

Let us consider the case $\rho = \frac{1}{32}$ so that both returns are positively related. We take $\pi = 0.4$, meaning that the reference portfolio $\overline{X}_{0.4}$ comprises 40% of unit wealth invested in asset 1 and we wonder whether this proportion should be increased. Figure 4.1 displays the curves $z \mapsto E\big[X_1 I[\overline{X}_{0.4} \leq z]\big]$ and $z \mapsto E\big[X_2 I[\overline{X}_{0.4} \leq z]\big]$. These are step functions exhibiting jumps at the four possible values of $\overline{X}_{0.4}$, that is 1.04, 1.12, 1.16, and 1.24. The two curves are at zero before 1.04 and at $E[X_1] = E[X_2] = 1.15$ after 1.24. We clearly see on Figure 4.1 that the two curves intersect so that condition (3.1) is violated and risk-averse investors do not unanimously agree to invest more that 40% of their initial wealth in asset 1. We could nevertheless wonder whether a subset of these decision makers would agree to do so.

This motivates the restriction of the set of investors to prudent ones. As prudence is usually justified by reference to the decision of building up precautionary savings in order to better face future income risk, this behavioral trait is quite natural in investment problems. Consider now prudent investors and define the subset $\mathcal{U}_{\text{ra-p}}$ of $\mathcal{U}_{\text{ra}}$ consisting in all $u \in \mathcal{U}_{\text{ra}}$ with third derivative u''' such that $u''' \geq 0$. The next result extends Proposition 3.1 to the case of prudent risk-averse decision makers. Precisely, it provides different conditions such that the optimal share $\lambda^\star$ invested in the first asset by every prudent risk-averse decision maker is at least equal to some given percentage. We denote as $x_+ = \max\{x, 0\}$ the positive part of x, i.e. $x_+ = x$ if $x > 0$ and $x_+ = 0$ otherwise.

Proposition 4.1. *Consider a fixed percentage $\pi \in [0, 1]$. The optimal share $\lambda^\star$ invested in the first asset is at least equal to π for every $u \in \mathcal{U}_{ra\text{-}p}$ if, and only if, $E[X_1] \geq E[X_2]$ and one of the following equivalent conditions is fulfilled:*

$$
\begin{aligned}
& E\big[X_1(z - \overline{X}_\pi)_+\big] \geq E\big[X_2(z - \overline{X}_\pi)_+\big] \text{ for all } z \in [a, b] && (4.2)\\
\Leftrightarrow\ & Cov\big[X_1 - X_2, (z - \overline{X}_\pi)_+\big] \geq E[X_2 - X_1]E[(z - \overline{X}_\pi)_+] \text{ for all } z \in [a, b]. && (4.3)
\end{aligned}
$$

Proof. Condition (3.3) still applies but with u' being now decreasing and convex. Let us use integration by parts in the two integrals appearing in formula (3.4) to get

$$
\begin{aligned}
E[X_1 u'(\overline{X}_\pi)] &= u'(b)E[X_1] - bu''(b)E[b - \overline{X}_\pi] + \int_a^b bu'''(z_2)E[(z_2 - \overline{X}_\pi)_+]dz_2 \\
&\quad + u''(b)E[(b - X_1)(b - \overline{X}_\pi)] - \int_a^b u'''(z_2)E[(b - X_1)(z_2 - \overline{X}_\pi)_+]dz_2.
\end{aligned}
$$

This gives

$$
\begin{aligned}
E[(X_1 - X_2)u'(\overline{X}_\pi)] &= u'(b)E[X_1 - X_2] + u''(b)E\big[(X_2 - X_1)(b - \overline{X}_\pi)\big] \\
&\quad + \int_a^b u'''(z_2)E[(X_1 - X_2)(z_2 - \overline{X}_\pi)_+]dz_2. \qquad (4.4)
\end{aligned}
$$

Hence, the proportion invested in asset 1 is at least π for every $u \in \mathcal{U}_{\text{ra-p}}$ if $E[X_1] \geq E[X_2]$ and (4.2) is fulfilled. To get the converse implication, notice that condition (3.3) with $u(x) = x$ ensures that $E[X_1] \geq E[X_2]$. Inserting the utility function $u(x) = -(z - x)_+^2$ in (3.3) shows that condition (4.2) must also hold. Finally, condition (4.3) easily follows as

$$
E[(X_1 - X_2)(z - \overline{X}_\pi)_+] = E[X_1 - X_2]E[(z - \overline{X}_\pi)_+] + Cov[X_1 - X_2, (z - \overline{X}_\pi)_+]
$$

and this ends the proof. □

Notice that compared to Proposition 3.1, we now need an additional condition imposed on the first moments of X_1 and X_2.

Let us come back to the introductory example of this section. Figure 4.2 displays the curves $z \mapsto E\big[X_1(z - \overline{X}_{0.4})_+\big]$ and $z \mapsto E\big[X_2(z - \overline{X}_{0.4})_+\big]$. We see that condition (4.2) is fulfilled in this case. Whereas risk-averse investors did not all agree to invest more than 40% of their initial wealth in asset 1, all the risk-averse and prudent ones among them agree about this decision.

If $E[X_1] = E[X_2]$ then Proposition 4.1 shows that $\lambda^\star \geq \pi$ for any $u \in \mathcal{U}_{\text{ra-p}}$ when

$$
Cov\big[X_1, (z - \overline{X}_\pi)_+\big] \geq Cov\big[X_2, (z - \overline{X}_\pi)_+\big] \text{ for all } z.
$$

In particular, we get for $z = b$

$$
Cov\big[X_1, \overline{X}_\pi\big] \leq Cov\big[X_2, \overline{X}_\pi\big].
$$

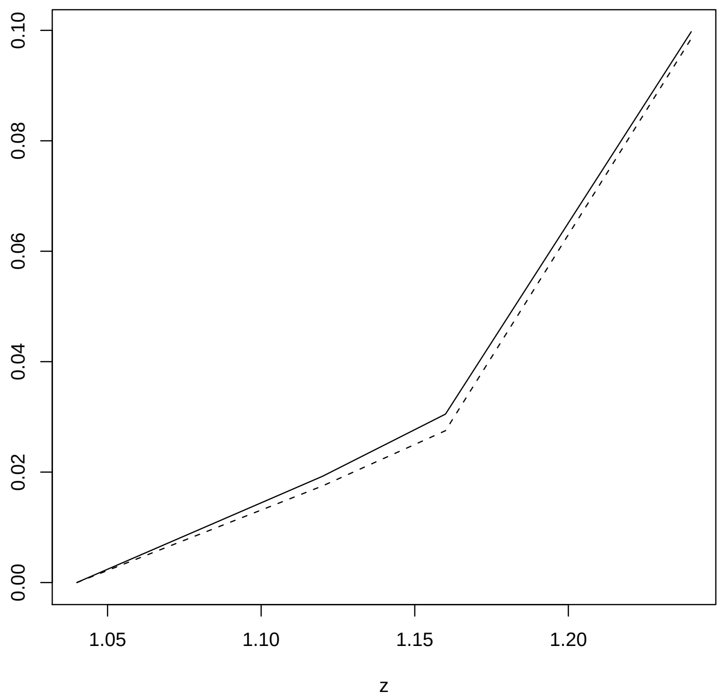

Figure 4.2: Graphs of $z \mapsto E\big[X_1(z-\overline{X}_\pi)_+\big]$ (continuous line —) and $z \mapsto E\big[X_2(z-\overline{X}_\pi)_+\big]$ (broken line - - -) for $\pi = 0.4$ when the joint distribution of asset returns X_1 and X_2 is given by (4.1) with $\rho = \frac{1}{32}$.

It is thus necessary (but not sufficient) for investing at least π in asset 1 that the covariance of X_1 with the reference portfolio $\overline{X}_\pi$ is smaller than that of X_2.

Notice that $(z - \overline{X}_\pi)_+$ is the payoff of a put option written on the performance of the reference portfolio $\overline{X}_\pi$, with exercise price z. If $E[X_1] = E[X_2]$ then the optimal proportion invested in asset 1 is larger than π if the covariance between X_1 and the payoff of this put option is always larger than the corresponding covariance with X_2. If X_1 and X_2 are identically distributed, or simply have the same variance, then the dominating asset in the portfolio is the one which is more correlated with the put option payoff on the performance of the portfolio $\overline{X}_\pi$.

Remark 4.2. We can relate (3.6) to conditions (4.2)-(4.3) in Proposition 4.1, noting that

$$\begin{aligned} \int_a^z \big(E[X_1] - E[X_1|\overline{X}_\pi \leq s]\big) P[\overline{X}_\pi \leq s] ds &= E[X_1]E[(z - \overline{X}_\pi)_+] - E[X_1(z - \overline{X}_\pi)_+] \\ &= -Cov[X_1, (z - \overline{X}_\pi)_+]. \end{aligned}$$

This also allows us to derive inequalities involving covariances, as integration by parts in formula (3.7) gives

$$\begin{aligned} Cov[X_1, g(\overline{X}_\pi)] &= \int_a^b g'(z)\big(E[X_1] - E[X_1|\overline{X}_\pi \leq z]\big) P[\overline{X}_\pi \leq z] dz \\ &= g'(b) Cov[X_1, \overline{X}_\pi] + \int_a^b g''(z) Cov[X_1, (z - \overline{X}_\pi)_+] dz. \end{aligned}$$

Therefore, provided $E[X_1] = E[X_2]$, conditions (4.2)-(4.3) ensure that $Cov[X_1, g(\overline{X}_\pi)] \geq Cov[X_2, g(\overline{X}_\pi)]$ for any decreasing and convex transformation g.

Example 4.3 (Positive demand)**.** The particular case $\pi = 0$ has been considered by Denuit et al. (2014) who extended the analysis conducted in Wright (1987) to higher-order risk attitudes.

Example 4.4 (Imprudent investors)**.** Consider two asset returns with the same means and variances, i.e. such that $E[X_1] = E[X_2]$ and $E[X_1^2] = E[X_2^2]$. Then, (4.4) with $\pi = \frac{1}{2}$ gives

$$E[(X_1 - X_2)u'(\overline{X}_\pi)] = \int_a^b u'''(z_2) E\left[(X_1 - X_2)\left(z_2 - \frac{X_1 + X_2}{2}\right)_+\right] dz_2.$$

If (4.2) holds true with $\pi = 0.5$ then we see that

(i) all prudent decision makers ($u''' \geq 0$) invest at least π in asset 1 as $\mathcal{O}'(\pi) \geq 0$.

(ii) all imprudent decision makers ($u''' \leq 0$) invest at most π in asset 1 as $\mathcal{O}'(\pi) \leq 0$.

Notice that decision makers with quadratic utilities ($u''' = 0$) are indifferent between the various portfolio compositions as $\mathcal{O}'(\pi) = 0$ for every proportion π in this case.

The results obtained for risk-averse and prudent investors can be extended to investors whose preferences exhibit risk apportionment up to any order in the sense of Eeckhoudt and Schlesinger (2006). This extension can be found in Appendix B. However, contrary to prudence, the conditions do not possess intuitive interpretations anymore.

5 Discussion

The notion of prudence is now well accepted in the economics literature, almost at parity with that of risk aversion. Besides its initial implications for the analysis of savings decision, it has been useful also to analyze other problems such as self-protection or optimal audits. Surprisingly however its implications for portfolio composition have not been analyzed so far and we have tried here to compensate for this deficiency.

The existing literature looks at the role of only risk aversion in the optimal composition of a portfolio of two possibly correlated risky assets. Thanks to an expansion formula presented in Appendix A, we have been able to summarize and extend the existing literature in a unified way. Then in Section 4, we have made the additional assumption that the decision maker is risk-averse and prudent. This additional requirement of prudence has lead to new results about diversification or about the 50% rule. Besides, when the two risky assets have the same mean, these conditions can be interpreted in terms of covariances with the payoffs of European put options written on the reference portfolio, replacing the digital options protecting against weak performances of this portfolio for risk-averse investors. An extension to higher-order risk apportionments has been proposed in Appendix B.

Acknowledgements

The financial support of PARC "Stochastic Modelling of Dependence" 2012-17 awarded by the Communauté française de Belgique is gratefully acknowledged by Michel Denuit.

References

- Chiu, W.H. (2005). Degree of downside risk aversion and self-protection. Insurance: Mathematics and Economics 36, 93-101.

- Clark, E., Jokung, O. (1999). A note on asset proportions, stochastic dominance, and the 50% rule. Management Science 45, 1724-1727.

- Denuit, M., Huang, R., Tzeng, L. (2014). Almost expectation and excess dependence notions. Theory and Decision, in press.

- Denuit, M., Rey, B. (2010). Prudence, temperance, edginess, and risk apportionment as decreasing sensitivity to detrimental changes. Mathematical Social Sciences 60, 137-143.

- Dionne, G., Li, J., Okou, C. (2012). An extension of the consumption-based CAPM model. Available at SSRN: http://ssrn.com/abstract=2018476.

- Drèze, J., Modigliani, F. (1972). Consumption decision under uncertainty. Journal of Economic Theory 5, 308-335.

- Eeckhoudt, L., Schlesinger, H. (2006). Putting risk in its proper place. American Economic Review 96, 280-289.

- Fagart, M.C., Sinclair-Desgagné, B. (2007). Ranking contingent monitoring systems. Management Science 53, 1501-1509.

- Hadar, J., Seo, T.K. (1988). Asset proportions in optimal portfolios. Review of Economic Studies 55, 459-468.

- Leland, H. (1968). Saving and uncertainty: The precautionary demand for saving. Quarterly Journal of Economics 82, 465-473.

- Kimball, M.S. (1990). Precautionary savings in the small and in the large. Econometrica 58, 53-73.

- Sandmo, A. (1970). The effect of uncertainty on saving decisions. Review of Economic Studies 37, 353-360.

- Shalit, H., Yitzhaki, S. (1994). Marginal conditional stochastic dominance. Management Science 40, 670-684.

- Wright, R. (1987). Expectation dependence of random variables, with an application in portfolio theory. Theory and Decision 22, 111-124.

- Yitzhaki, S., Olkin, I. (1991). Concentration indices and concentration curves. In "Stochastic Orders and Decision under Risk", AMS Lecture Notes-Monograph Series 19, 380-392.

Appendix

A A useful expansion formula

Consider two random variables Z_1 and Z_2 valued in $[a,b]$ and a real valued function g with domain $[a,b]\times[a,b]$. Let $g^{(i,j)}$ denote the (i,j)th partial derivative of g, i.e. $g^{(i,j)}(z_1,z_2)=\frac{\partial^{i+j}}{\partial z_1^i \partial z_2^j}g(z_1,z_2)$. Integration by parts shows that

$$\begin{aligned} E[g(Z_1,Z_2)] &= E[g(Z_1,b)] - \int_a^b g^{(0,1)}(b,z_2)P[Z_2\le z_2]dz_2 \\ &\quad + \int_a^b\int_a^b \Pr[Z_1\le z_1, Z_2\le z_2] g^{(1,1)}(z_1,z_2)dz_1dz_2. \end{aligned}$$

Integrating by parts the last double integral gives

$$\begin{aligned} &\int_a^b\int_a^b \Pr[Z_1\le z_1, Z_2\le z_2] g^{(1,1)}(z_1,z_2)dz_1dz_2 \\ =& \int_a^b g^{(1,1)}(b,z_2)\left(\int_a^b \Pr[Z_1\le \xi_1, Z_2\le z_2]d\xi_1\right)dz_2 \\ &-\int_a^b\int_a^b\left(\int_a^{z_1}\Pr[X_1\le\xi_1, X_2\le z_2]d\xi_1\right)g^{(2,1)}(z_1,z_2)dz_1dz_2. \end{aligned}$$

Now, as

$$\begin{aligned} \int_a^{z_1}\Pr[Z_1\le\xi_1, Z_2\le z_2]d\xi_1 &= \int_a^{x_1} E\Big[I[Z_1\le\xi_1]I[Z_2\le z_2]\Big]d\xi_1 \\ &= E\left[\int_a^{z_1} I[Z_1\le\xi_1]d\xi_1 I[Z_2\le z_2]\right] \\ &= E\Big[(z_1-Z_1)_+ I[Z_2\le z_2]\Big] \end{aligned}$$

the expectation $E[g(Z_1,Z_2)]$ can be expanded as follows:

$$\begin{aligned} E[g(Z_1,Z_2)] &= E[g(Z_1,b)] - \int_a^b g^{(0,1)}(b,z_2)P[Z_2\le z_2]dz_2 \\ &\quad + \int_a^b g^{(1,1)}(b,z_2)E\Big[(b-Z_1)I[Z_2\le z_2]\Big]dz_2 \\ &\quad - \int_a^b\int_a^b g^{(2,1)}(z_1,z_2)E\Big[(z_1-Z_1)_+I[Z_2\le z_2]\Big]dz_1dz_2. \end{aligned} \tag{A.1}$$

B Extension to higher-order risk attitudes

The results obtained for prudent investors can be extended to investors whose preferences exhibit risk apportionment of any order in the sense of Eeckhoudt and Schlesinger (2006).

Henceforth, wewrite $u^{(n)}$ for the nth derivative of u, $n = 1, 2, 3, 4, \ldots$; the notations u', u'', and u''' and $u^{(1)}$, $u^{(2)}$, and $u^{(3)}$, respectively, are used interchangeably. Recall that the preferences expressed by a differentiable utility function u satisfies risk apportionment of order n if it fulfills the condition $(-1)^{n+1}u^{(n)} \geq 0$. Prudence, temperance, and edginess respectively correspond to risk apportionment of order 3, 4, and 5.

Assume now that the risk-averse decision maker exhibits prudence and temperance, i.e. $(-1)^{n+1}u^{(n)} \geq 0$ holds for $n = 1, 2, 3, 4$. Integrating by parts (4.4) gives

$$\begin{aligned} E[(X_1 - X_2)u'(\overline{X}_\pi)] &= u'(b)E[X_1 - X_2] + u''(b)E\big[(X_2 - X_1)(b - \overline{X}_\pi)\big] \\ &\quad + u'''(b)E\left[(X_1 - X_2)\frac{(b - \overline{X}_\pi)^2}{2}\right] \\ &\quad + \int_a^b u^{(4)}(z_2)E\left[(X_2 - X_1)\frac{(z_2 - \overline{X}_\pi)^2}{2}\right] dz_2. \end{aligned} \tag{B.1}$$

Provided $E[X_1] \geq E[X_2]$, the analog of condition (4.2) for the subset of temperant investors becomes

$$E\big[X_1(b - \overline{X}_\pi)\big] \geq E\big[X_2(b - \overline{X}_\pi)\big] \text{ and } E\big[X_1(z - \overline{X}_\pi)_+^2\big] \geq E\big[X_2(z - \overline{X}_\pi)_+^2\big] \text{ for all } z.$$

Proceeding in the same way, we see that every investor whose preferences exhibit risk apportionment of orders 1 to n includes a proportion at least equal to π of asset 1 in his optimal portfolio when $E[X_1] \geq E[X_2]$ if

$$E\big[X_1(b - \overline{X}_\pi)^k\big] \geq E\big[X_2(b - \overline{X}_\pi)^k\big] \text{ for } k = 1, \ldots, n-3$$

and

$$E\big[X_1(z - \overline{X}_\pi)_+^{n-2}\big] \geq E\big[X_2(z - \overline{X}_\pi)_+^{n-2}\big] \text{ for all } z.$$

The minimum demand conditions are thus structured similarly, whatever the order n of risk apportionment, except that the shortfall $(z - \overline{X}_\pi)_+$ in the performances of the reference portfolio $\overline{X}_\pi$ used for prudent investors is replaced by its increasing powers.