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Parametrically guided nonparametric density and hazard estimation with censored data

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Abstract

The parametrically guided kernel smoother proposed by Hjort and Glad (1995) is a promising nonparametric estimation approach that aims to reduce the bias of the classical kernel density estimator without increasing its variance. In this paper we generalize this method to the censored data case and show how it can be used for density and hazard function estimation. The asymptotic properties of the proposed estimators are established and their performance is evaluated via finite sample simulations. The method is also applied to data coming from a study where one is interested in the time to return to drug use.

Key Words: Density estimation; Kaplan-Meier estimator; Kernel smoothing; Maximum likelihood; Right censoring.

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1 Introduction

Censored data appear in a broad variety of research studies with practical applications. Random right censoring is one of the most common types of censoring. For example in medical, economic or engineering studies, it frequently happens that the variable of interest T is only partially observed due to the earlier occurrence of a censoring event. In such studies, the estimation of the probability density and hazard function of T has received considerable attention in the literature, as it allows to visualize and explore the distribution of data.

In this paper we wish to estimate the density and hazard function when T is subject to right censoring, by using a hybrid estimation method that has at the same time nonparametric and parametric ingredients. These two extremal estimation approaches have rather opposite characteristics. The fully parametric approach is accurate and powerful when the parametric family is correctly chosen, otherwise it can lead to incorrect inference. The fully nonparametric approach includes several methods, among which the popular kernel smoothing procedure. It is very flexible, since it does not rely on any restrictive assumptions about the form of the underlying density or hazard function. However, the resulting estimator has typically a slower rate of convergence.

In the case where the data are not subject to censoring, there is a large variety of approaches to estimate the density and the hazard function that are either semiparametric or that use aspects from both the nonparametric and the parametric school, and that are hence situated in between these two extreme approaches. One of these approaches is the parametrically guided nonparametric estimator proposed by Hjort and Glad (1995). The basic idea of this approach is to start with any parametric density estimator and then to adjust this first stage parametric approximation using a nonparametric kernel-type estimator of a particular correction factor. More precisely, the key identity underlying the parametrically guided nonparametric approach is

$$f(t) = f_{\widehat{\theta}}(t) r_{\widehat{\theta}}(t),$$

where $r_{\widehat{\theta}}(t) = \frac{f(t)}{f_{\widehat{\theta}}(t)}$, $f_{\widehat{\theta}}(t)$ is a first stage parametric density approximation and $\widehat{\theta}$ is an estimator of the least false value θ^* according to a certain distance measure between $f(\cdot)$ and $f_\theta(\cdot)$ (see Assumption 3.3, below). Hjort and Glad (1995) defined the parametrically guided nonparametric estimator by

$$\widehat{f}_{\widehat{\theta}}(t) = f_{\widehat{\theta}}(t) \widehat{r}_{\widehat{\theta}}(t), \tag{1.1}$$

where $\widehat{r}_{\widehat{\theta}}(\cdot)$ is a kernel-type nonparametric estimator of the correction factor $r_{\widehat{\theta}}(\cdot)$. Essentially, this multiplicative correction does not affect the variance but can reduce the bias. The intuitive idea behind this approach is that if the parametric estimator $f_{\widehat{\theta}}(\cdot)$ is close to the true density $f(\cdot)$, the multiplicative correction function $r_{\widehat{\theta}}(\cdot)$ will be smoother than the true density $f(\cdot)$ and therefore simpler to estimate using kernel smoothing, resulting in an improved $\widehat{f}_{\widehat{\theta}}(\cdot)$ compared to the traditional kernel estimator. If the true density is far from the parametric estimator, then there is not much loss in accuracy for the parametrically guided nonparametric estimator.

The aim of this paper is to extend their method to the case of censored data. To the best of our knowledge, except for the recent work of Talamakrouni *et al.* (2014), who studied a guided local linear estimator of a regression function when the response is subject to censoring, the parametrically guided nonparametric method has never been investigated in the context of censored data. In addition to studying the estimation of the density function, we also propose and study a parametrically guided nonparametric estimator of the hazard rate function in the presence of censoring.

Apart from the above parametrically guided nonparametric estimator of Hjort and Glad (1995), there have been other proposals in the literature that combine the nice features of both the parametric and the nonparametric approach. These methods are quite different but can also achieve bias reduction compared to the fully nonparametric method. As far as we are aware of, except for the paper of Copas (1995) who adapted a local maximum likelihood estimator to censored data, none of them has been considered so far in the context of censored data. First of all, we find the projection pursuit density estimation developed by Friedman *et al.* (1984) for a multivariate density using a similar multiplicative correction. Hjort (1986) and Buckland (1992) introduced similar ideas using an estimated orthogonal expansion for the multiplicative correction factor. Hjort and Jones (1996) proposed a local parametric density estimator based on a local kernel smoothed likelihood function. This approach has a similar intention as the approach of Copas (1995) but is somehow more general. Another class of local likelihood methods has been discussed by Eguchi and Copas (1998). Efron and Tibshirani (1996) combined the maximum likelihood and the kernel estimator by putting an exponential family through a kernel estimator. Other semiparametric estimators involving an extra parameter have been proposed in the literature. For example, Olkin and Spiegelman (1987) and Faraway (1989) considered a convex combination of a parametric and a nonparametric estimate, and afterwards, Naito (2004) constructed a class of semi-parametric estimators using a local L_2-fitting criterion to estimate the correction factor. Finally, more recently, Veraverbeke *et al.* (2014) discussed a parametrically pre-adjusted nonparametric density estimator.

The paper is organized as follows. The next section explains in detail the proposed methodology. Section 3 provides some asymptotic results for the proposed estimators, while Section 4 investigates the finite sample properties of the new estimators. In Section 5 we apply the proposed method to data on the time to return to drug use from a study of the AIDS research unit of the University of Massachusetts. Finally, some general conclusions are drawn in Section 6. The proofs are collected in the Appendix.

2 Methodology

Let T be a variable of interest with density f and distribution function F, and let C be a censoring variable with continuous distribution function G. We assume throughout our paper that T is independent of C. Under random right censoring, the variable T is not completely observed. One can only observe (X, δ), where $X = \min(T, C)$, $\delta = I(T \leq C)$ and $I(\cdot)$ is the indicator function. Our first objective is to estimate the probability density function f using the observed i.i.d sample $(X_i, \delta_i), i = 1, \ldots, n$ of (X, δ).

The kernel-based density estimator that we are currently investigating has been extended to censored data by Blum and Susarla (1980), among others. The estimator is based on the Kaplan-Meier (1958) estimator $\widehat{F}$ of the distribution function F and is defined as follows:

$$\widehat{f}(t) = \frac{1}{h} \int_{-\infty}^{+\infty} K\Big(\frac{t-s}{h}\Big) d\widehat{F}(s), \tag{2.2}$$

where K is a kernel function, $0 < h \equiv h_n$ is a bandwidth and $\widehat{F}(t)$ is defined by (in the absence of ties)

$$\widehat{F}(t) = 1 - \prod_{i:X_i \leq t} \Big(1 - \frac{1}{\sum_{j=1}^{n} \mathbf{1}_{\{X_j \geq X_i\}}}\Big)^{\delta_i}. \tag{2.3}$$

We also aim to estimate the hazard function $\lambda(\cdot)$ defined by $\lambda(t) = f(t)/(1 - F(t))$. A natural nonparametric estimator for the hazard function can be formed by dividing the kernel density estimator by the Kaplan-Meier estimator of the survival function $1 - F(\cdot)$:

$$\widehat{\lambda}(t) = \widehat{f}(t)/1 - \widehat{F}(t).$$

In this framework, the properties of the kernel density and hazard estimators have been studied by Blum and Susarla (1980), Földes *et al.* (1981), Tanner and Wong (1983), Padgett and McNichols (1984), Mielniczuk (1986), Lo *et al.* (1989), Xiang (1994) and Giné and Guillou (2001), among others.

Note that, the kernel estimators defined above are by construction completely nonparametric. In the uncensored data context, Hjort and Glad (1995) proposed a parametrically guided kernel density estimator (PGK) as an alternative to the traditional kernel density estimator (TK). As argued in the introduction, the PGK estimator combines the advantages of both parametric and nonparametric approaches and includes a prior information that allows the bias reduction of the PGK estimator compared to the TK estimator.

For censored data, we propose to multiply the first stage parametric estimator $f_{\widehat{\theta}}(t)$ in expression (1.1) with the following nonparametric kernel-type estimator of the correction function $r_{\widehat{\theta}}(t)$ adapted to censored data:

$$\widehat{r}_{\widehat{\theta}}(t) = \frac{1}{h}\int_{-\infty}^{+\infty} K\Big(\frac{t-s}{h}\Big)\frac{1}{f_{\widehat{\theta}}(s)}d\widehat{F}(s).$$

The ensuing PGK density estimator is

$$\begin{aligned}\widehat{f}_{\widehat{\theta}}(t) &= \frac{1}{h}\int_{-\infty}^{+\infty} K\Big(\frac{t-s}{h}\Big)\frac{f_{\widehat{\theta}}(t)}{f_{\widehat{\theta}}(s)}d\widehat{F}(s)\\ &= \frac{1}{h}\sum_{i=1}^{n} K\Big(\frac{t-X_{(i)}}{h}\Big)\frac{f_{\widehat{\theta}}(t)}{f_{\widehat{\theta}}(X_{(i)})}W_{(i)}, \qquad (2.4)\end{aligned}$$

where $X_{(i)}$ is the i^{th} order statistics of the X's and $W_{(i)}$ is the size of the jump of $\widehat{F}$ at $X_{(i)}$. Note that when the data are completely observed, the weights $W_{(i)}$ are all equal to $1/n$ and the PGK estimator given above reduces to the estimator defined by Hjort and Glad (1995). Naturally the PGK estimator that we propose for the hazard function $\lambda(\cdot)$ is

$$\widehat{\lambda}_{\widehat{\theta}}(t) = \widehat{f}_{\widehat{\theta}}(t)/(1-\widehat{F}(t)). \qquad (2.5)$$

As we will see in the following section, the multiplicative correction used in the PGK density and hazard function estimators does not affect the variance but can reduce the bias compared to the traditional kernel estimators defined above.

3 Asymptotic results

This section is devoted to the development of the asymptotic normality of the PGK estimators $\widehat{f}_{\widehat{\theta}}(\cdot)$ and $\widehat{\lambda}_{\widehat{\theta}}(\cdot)$. For the PGK density estimator, we split the problem into two parts. First, we establish in Theorem 3.1 the asymptotic normality of $\widehat{f}_*(\cdot)$, an estimator of $f(\cdot)$ based on a given non-random guide $f_*(\cdot)$. Then, in Theorem 3.2, we extend this result to the case of a data-driven guide. Finally, in Theorem 3.3 we prove the asymptotic normality of the PGK estimator of the hazard function $\widehat{\lambda}_{\widehat{\theta}}(\cdot)$.

As stated in the previous section, under random right censoring the PGK estimator depends on the Kaplan-Meier estimator $\widehat{F}$, which is defined as a product (see expression (2.3)). This adds some extra complexity to the PGK estimation approach compared to the uncensored case. To circumvent these technical difficulties we mainly use the asymptotic i.i.d. representation of the Kaplan-Meier estimator investigated in Lo *et al.* (1989).

Let $\tau < \tau_H$, where $\tau_H = \sup\{t : H(t) < 1\}$ is the right endpoint of the distribution function $H(t) = P(X \leq t)$. Also, let $H_1(t) = P(X \leq t, \delta = 1)$, and define $\mu_K^2 = \int u^2 K(u)du$. The kernel function $K : \mathbb{R} \to \mathbb{R}$, the bandwidth h and the density $f(\cdot)$ are assumed to satisfy the following conditions for a fixed value $t \leq \tau$.

Assumption 3.1.

(A.1) The kernel K is a symmetric, continuously differentiable probability density function with compact support $[-1, 1]$.

(A.2) The bandwidth sequence h satisfies $h \to 0$ and $nh^2(\log n)^{-2} \to \infty$.

(A.3) f is twice continuously differentiable in a neighborhood of t and $f(t) > 0$.

3.1 Guided kernel density estimator with a fixed guide

Let $f_*(t)$ be a non-random density function that approximates $f(t)$, and let $\widehat{f}_*(t)$ be the corresponding PGK estimator defined as

$$\widehat{f}_*(t) = \frac{1}{h} \int_{-\infty}^{+\infty} K\left(\frac{t-s}{h}\right) \frac{f_*(t)}{f_*(s)} d\widehat{F}(s). \tag{3.6}$$

In the next section we will replace $f_*(\cdot)$ by the best approximation of $f(\cdot)$ within a certain parametric class, but for the time being $f_*(\cdot)$ can be any deterministic density.

Note that if $f_*(t)$ would be a uniform density, then $\widehat{f}_*(t)$ reduces to the TK estimator, which means that the PGK estimator is a generalization of the TK estimator. The following additional conditions are required for a fixed point $t \leq \tau$ at which we want to estimate the density.

Assumption 3.2.

(B.1) The density $f_*(\cdot)$ is twice continuously differentiable in a neighborhood of t.

(B.2) The density $f_*(\cdot)$ satisfies $f_*(t) > 0$.

The following theorem provides the asymptotic distribution of the PGK estimator $\widehat{f}_*(\cdot)$ using a non-random guide.

Theorem 3.1. *Suppose Assumptions 3.1 and 3.2 hold.*

1. *Then,*

$$\widehat{f}_*(t) - f(t) = \frac{1}{nh}\sum_{i=1}^{n} U_{in}(t) + \frac{1}{h}\int_{-\infty}^{+\infty} K\left(\frac{t-s}{h}\right)\frac{f_*(t)}{f_*(s)}dF(s) - f(t) + O_p(n^{-1/2}),$$

where

$$U_{in}(t) = \int_{-1}^{1} \xi_i(t-uh)K'(u)du,$$

$$\xi_i(t) = \int_{-\infty}^{X_i \wedge t} \frac{dH_1(s)}{(1-H(s))^2} + \frac{I\{X_i \le t, \delta_i = 1\}}{1-H(X_i)}.$$

2. *Moreover,*

$$\sqrt{nh}\Big(\widehat{f}_*(t) - f(t) - B_*(t) + o(h^2)\Big) \xrightarrow{d} \mathcal{N}\Big(0, \sigma^2(t)\Big), \tag{3.7}$$

where

$$B_*(t) = \frac{1}{2}h^2\mu_K^2 r_*''(t) f_*(t),$$

$r_*(t) = f(t)/f_*(t)$ *and* $\sigma^2(t) = [f(t)/(1-G(t))]\int_{-1}^{1} K^2(u)du.$

Note that the choice of the guide has an obvious impact on the expression of the asymptotic bias $B_*(t)$, whilst the variance $\sigma^2(t)$ is invariant under this choice and is the same as for the TK estimator.

3.2 Guided kernel density estimator with an estimated guide

In this section, we investigate the situation where the guide is derived from the data by a first stage estimation procedure. We consider a possibly misspecified parametric model $\{f_\theta(\cdot) : \theta \in \Theta\}$ and assume that there exists an estimator $\widehat{\theta}$ that converges in probability to a finite limit θ^*. We need the following additional conditions for a fixed point $t \le \tau$.

Assumption 3.3.

(C.1) The parametric density function f_θ belongs to a parametrically indexed class defined by the following characteristics:

1. *$\theta \in \Theta$, where Θ is a compact subset of $\mathbb{R}^p$.*
2. *The function $(t, \theta) \mapsto f_\theta(t)$ is twice continuously differentiable with respect to t and the components of θ in a neighborhood of t and θ^*.*

(C.2) The parameter $\theta^* \in \Theta$ satisfies the following conditions:

1. *$\widehat{\theta} - \theta^* = O_p(n^{-1/2})$.*
2. *The density $f_{\theta^*}(\cdot)$ satisfies $f_{\theta^*}(t) > 0$.*

In order to be as general as possible, in this paper, we don't restrict ourselves to a particular parametric estimation procedure. However, to illustrate the idea and give an example of an estimator that satisfies the conditions above, especially assumption (C.2.1), we discuss now the case of the maximum likelihood estimator (MLE). Define

$$\theta^* = \arg\max_{\theta\in\Theta} \int_{-\infty}^{+\infty} \log(f_\theta(t)) dF(t).$$

This is the minimizer of $KL(f, f_\theta) = \int \log \frac{f(t)}{f_\theta(t)}\, dF(t)$, the Kullback-Leibler distance measure between the true density f and the parametric density model f_θ. If the parametric model is correct, i.e. if there exists a $\theta_0 \in \Theta$ such that $f(\cdot) = f_{\theta_0}(\cdot)$, then $\theta^* = \theta_0$. In the uncensored case, it is well known that the usual MLE given by $\widehat{\theta} = \arg\max_{\theta\in\Theta} n^{-1} \sum_i \log(f_\theta(X_i))$ is consistent for θ^*, even under misspecification. In the censored case, the analogue of $\widehat{\theta}$ is the approximate maximum likelihood estimator (AMLE) proposed by Oakes (1986) and defined as

$$\widehat{\theta} = \arg\max_{\theta\in\Theta} \int_{-\infty}^{+\infty} \log(f_\theta(t))\, d\widehat{F}(t),$$

where $\widehat{F}(\cdot)$ is the Kaplan-Meier estimator. Note that, in the uncensored case, the Kaplan-Meier estimator coincides with the empirical distribution function and therefore the AMLE reduces to the MLE. The properties of the AMLE estimator have been investigated by Suzukawa *et al.* (2001). Assuming that $\tau_F := \sup\{t : F(t) < 1\} \leq \tau_G := \sup\{t : G(t) < 1\}$ and under certain regularity assumptions, the authors prove that $\widehat{\theta}$ is $\sqrt{n}-$consistent.

Remark 3.1. *Even if the data are censored, the usual MLE estimator $\widehat{\theta}$ can still be used. However, under misspecification, this estimator does not convergence to θ^* but to another quantity; see Suzukawa et al. (2001) for more details.*

The following theorem is the most important result of the paper. It establishes that the PGK estimator with an estimated guide $f_{\widehat{\theta}}(t)$ is asymptotically equivalent to the PGK estimator with the fixed guide $f_{\theta^*}(t)$.

Theorem 3.2. *Suppose Assumptions 3.1 and 3.3 hold. Then,*

$$\sqrt{nh}\Big(\widehat{f}_{\widehat{\theta}}(t) - f(t) - B_{\theta^*}(t) + o(h^2)\Big) \xrightarrow{d} \mathcal{N}\Big(0, \sigma^2(t)\Big),$$

where $B_{\theta^*}(t) = \frac{1}{2}h^2\mu_K^2 r''_{\theta^*}(t) f_{\theta*}(t)$ *and* $r_{\theta^*}(t) = f(t)/f_{\theta*}(t)$.

First, notice that the expression of the asymptotic variance is independent of the parametric estimating procedure and is equal to that of the TK estimator. As revealed in the previous section by Theorem 3.1, the main difference between the behavior of the PGK estimator and the TK estimator appears in the term of the asymptotic bias $B_{\theta^*}(t)$, which depends on the parametric guide. Remind that, if the parametric guide is the uniform density, then the PGK estimator becomes the traditional kernel density estimator and $B_{\theta^*}(t)$ coincides with $B(t) = \frac{1}{2}h^2\mu_K^2 f''(t)$, the asymptotic bias of the TK estimator; see, for example, Lo *et al.* (1989). So, with an appropriate choice of the guide, i.e. when $|r''_{\theta^*}(t) f_{\theta*}(t)| \leq |f''(t)|$, the bias of the PGK estimator will be reduced in absolute value compared to that of the TK estimator, whilst the variance remains unchanged. If the parametric density is a good guess, then the correction function $r_{\theta^*}(\cdot)$ will be nearly constant and its second derivative $r''_{\theta^*}(\cdot)$ should be very small. In this case the bias reduction will be attained. Finally, in the ideal case when the parametric guide coincides with the true density we have that $B_{\theta^*}(t) = 0$. In such a case, one can choose an arbitrary large bandwidth to reduce the variance to its minimal possible value.

Remark 3.2. *In practice, the choice of the bandwidth is a crucial issue in kernel-based density estimation. The theoretical optimal bandwidth that minimizes the asymptotic mean integrated squared error (MISE) criterion is given by*

$$h_{opt} = \left(\frac{\int \sigma^2(t)dt}{\mu_K^4 \int (r''_{\theta^*}(t) f_{\theta*}(t))^2 dt}\right)^{1/5} n^{-1/5}.$$

This expression can hardly be used in practice, since it depends on many unknown components. To select the bandwidth h in our case, one can use for example the least squares cross validation *method or the bootstrap method discussed in Sánchez-Sellero et al. (1999). In our data analysis, see Section 5, we adopt the cross validation method as discussed in Wang and Wang (2007).*

We also point out that, the PGK method will work even if the parametric guide is not optimally chosen. However, an optimal choice of the parametric guide will improve the quality of the PGK estimator. One can for example use goodness-of-fit tests to choose the parametric guide; see for example Castro-Kuriss (2011).

3.3 Guided kernel hazard estimator with an estimated guide

The hazard function $\lambda(\cdot)$ has been extensively studied in the literature. The estimation by means of kernel methods has been investigated by Gefeller and Dette (1992), Gefeller and Michels (1992), Patil (1993), Müller and Wang (1994) and González-Manteiga *et al.* (1996), among others. The PGK estimator for the hazard function $\lambda(\cdot)$ that we proposed in the previous section is

$$\widehat{\lambda}_{\widehat{\theta}}(t) = \widehat{f}_{\widehat{\theta}}(t)/[1 - \widehat{F}(t)],$$

where $\widehat{f}_{\widehat{\theta}}(t)$ is the PGK density estimator given in (2.4) and $\widehat{F}(t)$ is the Kaplan-Meier estimator. Note that one can also replace the Kaplan-Meier estimator by a parametrically guided nonparametric version of the distribution function F or by any other estimator that has parametric and nonparametric ingredients (see e.g. Veraverbeke *et al.* (2014), Section 7.1, for an overview of possible estimators). However, given that the rate of convergence of the estimator of $F(t)$ will always be faster than the rate of convergence of the density estimator $\widehat{f}_{\widehat{\theta}}(t)$, the choice of the estimator of $F(t)$ has no impact on the asymptotic distribution of the estimator of $\lambda(t)$. For simplicity we therefore estimate $F(t)$ by the Kaplan-Meier estimator $\widehat{F}(t)$.

The following theorem deals with the asymptotic normality of the PGK hazard rate estimator.

Theorem 3.3. *Suppose Assumptions 3.1 and 3.3 hold. Then,*

$$\sqrt{nh}\Big(\widehat{\lambda}_{\widehat{\theta}}(t) - \lambda(t) - \beta_{\theta^*}(t) + o(h^2)\Big) \xrightarrow{d} \mathcal{N}\Big(0, \tau^2(t)\Big),$$

where

$$\beta_{\theta^*}(t) = \frac{1}{2}h^2\mu_K^2 r''_{\theta^*}(t) f_{\theta^*}(t)/[1 - F(t)],$$

and $\tau^2(t) = [\lambda(t)/(1 - H(t))]\int K^2(u)du$.

As for the density estimator, the asymptotic bias of the PGK hazard rate estimator depends on the parametric guide, while the asymptotic variance remains unchanged compared to the TK hazard estimator.

4 Simulation results

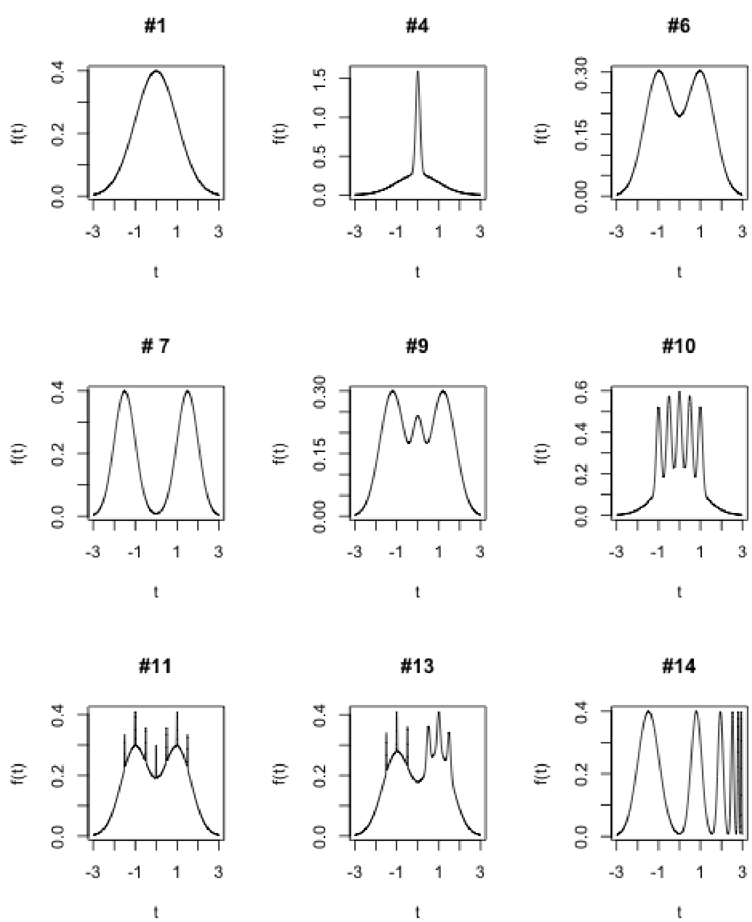
In this section we evaluate the finite sample performance of the PGK estimator by means of Monte Carlo simulations. To check the theoretical results and compare the PGK estimator with the TK estimator we investigate two examples. In the first example we study the classical class of normal mixture densities of Marron and Wand (1992), and in the second example we investigate the Weibull density. Along the simulations we consider the Epanechnikov kernel function K, and, for every estimator, we only show the results corresponding to the optimal bandwidth, i.e. the one that minimizes the empirical mean squared error (MSE), both for the PGK and the TK estimator.

4.1 Normal mixture model

The class of normal mixture densities of Marron and Wand (1992) includes fifteen densities that cover a broad variety of shapes. In the context of uncensored guided density estimation, this class was investigated by Hjort and Glad (1994) and Naito (2004), among others. We studied all the fifteen densities but for the sake of brevity we only show the results of the following ones: the normal density, #1, the kurtotic unimodal density, #4, the bimodal density, #6, the separated bimodal density, #7, the trimodal density, #9, the claw density, #10, the double claw density, #11, the asymmetric double claw density, #13, and the smooth comb density, #14. See Figure 1 for a plot of all these densities. In each case, independent and identically distributed variables T_i, $i = 1, \ldots, 400$, are drawn. Independently, we drawn the censoring variables C_i from the same distribution. This leads to 50% rate of censoring. As for the parametric guide, we consider a standard normal density whose parameters are estimated by maximum likelihood. So the first case (#1) is the only situation where the guide is correctly specified. We compute the PGK and the TK density estimators at $t = 0$ taking 100 equally spaced bandwidths over the interval $[0.01, 4]$. The squared bias ($\text{Bias}^2 \times 10^4$), the variance ($\text{Var} \times 10^4$) and the empirical mean squared error ($\text{MSE} \times 10^4$) of each estimator were computed. Table 1 provides the results with 1000 replications for cases #1, #4, #6, #7, #9#10, #11, #13, and #14. As already mentioned,

#1 corresponds to the case where the guide is a good guess of the true density. As expected, in this case, the bias of the PGK estimator is almost zero and is substantially reduced compared to that of the TK estimator. For cases #9 and #11 the PGK estimator is significantly better than the TK estimator. For cases #4 and #10 the MSE becomes very large and reveals unstable behavior for both estimators. In addition, although the MSE is not greatly enhanced, the bias and the MSE of the PGK estimator remain the smallest. Cases #7 and #14 show a quite similar behavior of the PGK and the TK estimators. However, even if the bias is still reduced, the TK estimator beats the PGK estimator in terms of the MSE, for cases #6 and #13. Finally, we point out that the selected bandwidths for both competitors were the same for most of the non-normal situations.

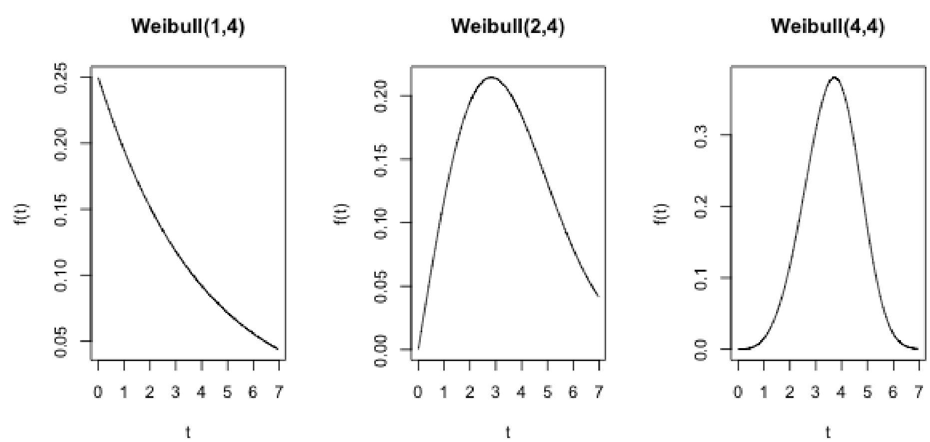
Figure 1: Standard normal and normal mixture target densities.



4.2 Weibull density with an exponential guide

In this model, the variable of interest T is generated from a Weibull distribution with a scale parameter $b = 4$ and a shape parameter taking three values $a = 1, 2, 4$. The graphs of the resulting densities are plotted in Figure 2. The censoring variable is also drawn from a Weibull distribution with shape parameter a and scale parameter given by $b((1-p)/p)^{1/a}$, ensuring a degree of censoring equal to p. We consider two censoring rates $p = 10\%$ and $p = 40\%$, and two sample sizes $n = 150$ and $n = 400$. As a parametric guide we use the exponential density $f_\theta(t) = \theta \exp(-\theta t)$, where θ is estimated using the approximated maximum likelihood estimator given by $\hat{\theta} = 1/\sum_{i=1}^{n} W_{(i)} X_{(i)}$, where $X_{(i)}$ are the ordered values of the observed variables $X_i = \min(T_i, C_i)$ and $W_{(i)}$ is the size of the jump of the Kaplan-Meier estimator at $X_{(i)}$ (see Suzukawa *et al.* (2001)). Note that the case $a = 1$ is the only situation where the guide is correctly specified. If $a \neq 1$ then the parametric guide is incorrect and deviates gradually from the true density. Our goal is to compare the performance of the PGK estimator with that of the TK estimator for both the density and the hazard function. To this end, we run 1000 simulations and for every generated data set we calculate the estimators at the point $t = 3$. We look for the optimal bandwidths via a grid on $[0.1, 5]$. The results are summarized in Tables 2 and 3 for the density and the hazard function, respectively.

Figure 2: Weibull density with shape parameters $a = 1, 2, 4$ and scale parameter $b = 4$.



We start with the simulation results for the density estimators. As expected, with a correct parametric guide ($a = 1$) we get the best results for the PGK estimator, the bias of the

PGK estimator is significantly reduced compared to that of the TK estimator. Regarding the MSE, the MSE is also reduced for the PGK estimator compared to that of the TK estimator except for the case with sample size 150 and censoring rate 40%, where we observe a slightly larger variance and MSE for the PGK estimator, and this is corrected with a larger sample size $n = 400$. For $a = 2$ and $a = 4$, even if the parametric guide is incorrect, the PGK remains significantly better than the TK estimator. Regarding the variance, as expected, the estimators have similar behavior except for the case $a = 4$ where the TK estimator has a larger variance. Generally, the MSE is not substantially reduced, because in this example, the variance dominates the bias.

For the hazard function, we computed the PKG and the TK estimators using the same data generating procedure as for density function. The results are summarized in Table 3 and show that the PGK estimator outperforms the TK estimator even if the parametric guide is not correctly specified. Note that for both density and hazard functions, increasing the sample size enhances the performance of the PGK estimator. Another point to remark is that, for the density and the hazard function, the selected optimal bandwidths for the PGK and the TK estimators are often close. Finally, we compared the PGK estimator based on the MLE and the PGK estimator based on the AMLE. Simulations not given here show that when the guide is correct the PGK estimator based on the MLE outperforms the PGK estimator based on the AMLE. This can be expected since in this case the MLE is consistent, while the PGK estimator based on the AMLE behaves better when the guide is misspecified (see Remark 3.2 and Suzukawa *et al.* (2001)).

5 Data Analysis

In this section, we use the PGK estimator for the analysis of the UIS dataset from the University of Massachusetts Aids Research Unit (UMARU) IMPACT Study. The variable of interest is the time in days to return to drug use (measured from admission) of a participant. Among a total of 628 observations, there are 120 censored observations which corresponds to 19.12% rate of censoring. The data with a detailed description of the study can be found in Section 1.3 of Hosmer *et al.* (2008). We use two different guides, an exponential density and a Weibull density for which the parameters are estimated using maximum likelihood. We use the Epanechnikov kernel and the choice of the bandwidths is achieved by a data driven bandwidth selection based on the least squared cross-validation method adapted to each estimator (see for example Wang and Wang (2007)). The selected bandwidths are: $h = 671.21$ for the PGK estimator with an exponential guide, $h = 716.66$ for the PGK estimator with a Weibull guide and $h = 663.63$ for the TK estimator.

The plots of the different estimators are given for density and hazard function respectively in Figures 3 and 4. We cannot know which estimator is better, but given the results of our simulation study, we guess that the TK method has tendency to either overestimating or underestimating the density and the hazard function and so the real risks to return to drug use. Finally, it seems that our PGK density estimators remove a considerable part of the boundary effect on the left endpoint of the distribution compared to the TK density estimator.

Figure 3: TK estimator and PGK estimators using exponential and Weibull guides of the density for the

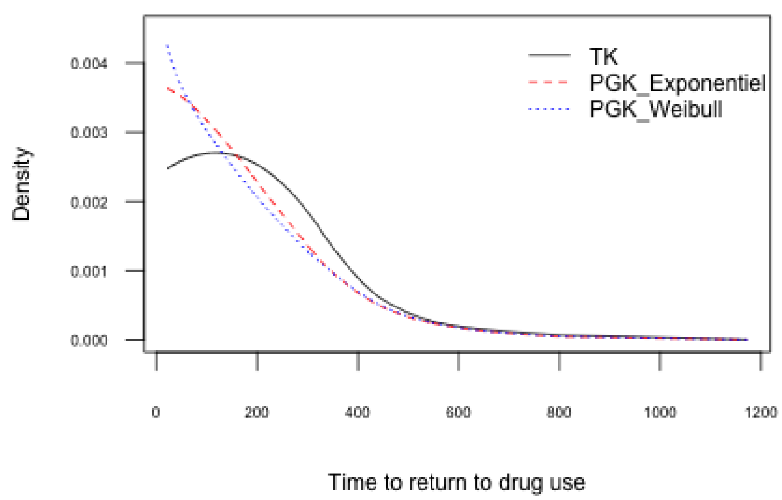
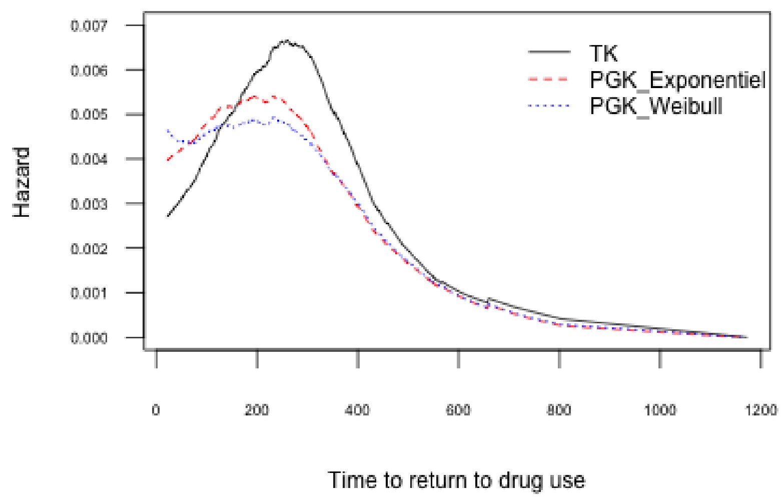


Figure 4: TK estimator and PGK estimators using exponential and Weibull guides of the hazard function



6 Conclusion

In this paper, we extended the parametrically guided kernel density and hazard estimators to the censored data framework. The proposed estimators are obtained by multiplying an initial parametric estimator by a nonparametric kernel type estimator of a certain correction function. We established the asymptotic normality of the proposed estimators and obtained asymptotic expressions of the bias and variance. Under certain regularity conditions, we proved that the bias of the proposed estimator can be reduced compared to that of the traditional kernel estimator, while the variance does not change. Simulations confirmed the theoretical results and provide the following remarks for the density and the hazard functions: the PGK estimator with censored data outperforms the TK estimator if the parametric guide is equal or close to the true target function and performs as the TK estimator if the parametric guide is misspecified. The application of the PGK estimator to the UIS dataset reveals that, in addition to bias reduction, the estimator also seems to correct in an automatic way for possible boundary effects. As pointed out by Hjort and Glad (1995) in the uncensored case, the advantages of the multiplicative PGK method come with some drawback caused by the correction factor $f_{\widehat{\theta}}(t)/f_{\widehat{\theta}}(X_{(i)})$, see equation (2.4). Small values of $f_{\widehat{\theta}}(X_{(i)})$ may affect the numerical stability of the resulting estimator especially with a "large" bandwidth and this affects also the MSE. One may correct for this by adding a small ϵ to both the numerator and the denominator or by adopting an additive parametric correction (instead of a multiplicative one). This method is under investigation and will be the subject of a future publication.

7 Appendix

Proof of Theorem 3.1. The PGK density estimator based on the non random parametric guide $f_*(\cdot)$ can be decomposed as follows:

$$\widehat{f}_*(t) - f(t) = (\widehat{f}_*(t) - \widetilde{f}(t)) + (\widetilde{f}(t) - f(t)), \tag{7.8}$$

where

$$\widetilde{f}(t) = \frac{1}{h}\int_{-\infty}^{+\infty} K\left(\frac{t-s}{h}\right)\frac{f_*(t)}{f_*(s)}dF(s).$$

1. For $t \le \tau$, we have

$$\begin{aligned}\widehat{f}_*(t) - \widetilde{f}(t) &= \frac{f_*(t)}{h}\int_{t-h}^{t+h} K\left(\frac{t-s}{h}\right)\frac{1}{f_*(s)}d(\widehat{F}(s) - F(s)) \\ &= \frac{f_*(t)}{h}\int_{-1}^{1}(\widehat{F}(t-uh) - F(t-uh))d\big(K(u)/f_*(t-uh)\big) \\ &= \frac{f_*(t)}{h}\int_{-1}^{1}(\widehat{F}(t-uh) - F(t-uh))\frac{K'(u)}{f_*(t-uh)}du \\ &\quad + f_*(t)\int_{-1}^{1}(\widehat{F}(t-uh) - F(t-uh))\frac{K(u)f_*'(t-uh)}{f_*^2(t-uh)}du \\ &= A_{1,n} + A_{2,n}.\end{aligned}$$

First, write

$$\begin{aligned}A_{1,n} &= \frac{1}{h}\int_{-1}^{1}(\widehat{F}(t-uh) - F(t-uh))K'(u)du \\ &\quad + \frac{f_*(t)}{h}\int_{-1}^{1}(\widehat{F}(t-uh) - F(t-uh))\left(\frac{K'(u)}{f_*(t-uh)} - \frac{K'(u)}{f_*(t)}\right)du \\ &= A_{11,n} + A_{12,n}.\end{aligned}$$

We start with $A_{12,n}$. We have

$$\begin{aligned} A_{12,n} &= \frac{1}{h}\int_{-1}^{1}(\widehat{F}(t-uh)-F(t-uh))\frac{f_*(t)-f_*(t-uh)}{f_*(t-uh)}K'(u)du \\ &= \frac{1}{h}\int_{-1}^{1}(\widehat{F}(t-uh)-F(t-uh))\frac{f_*'(t+\rho)uh}{f_*(t-uh)}K'(u)du, \end{aligned}$$

for some ρ between 0 and $-uh$. Therefore,

$$|A_{12,n}| \leq \sup_{s\in\aleph_t}|\widehat{F}(s)-F(s)|\sup_{s\in\aleph_t}|f_*'(s)|(\inf_{s\in\aleph_t} f_*(s))^{-1}\int_{-1}^{1}|K'(u)||u|du,$$

where $\aleph_t$ is a small neighborhood around t. Hence, under assumptions 3.1 and 3.2, and using the uniform rate of the Kaplan-Meier estimator (see e.g. Theorem 1 in Lo and Singh (1986)) we have that $A_{12,n} = O_p(n^{-1/2})$. Now, we treat the term $A_{11,n}$. We consider the i.i.d. decomposition of $\widehat{F}$ given in Lemma 2.1 in Lo *et al.* (1989):

$$\widehat{F}(s) - F(s) = n^{-1}\sum_{i=1}^{n}\xi_i(s) + r_n(s),$$

where $\xi_i(s) = -\int_{-\infty}^{X_i\wedge s}\big(1-H(x)\big)^{-2}dH_1(x) + \big(1-H(X_i)\big)^{-1}I\{X_i \leq s, \delta_i = 1\}$ and $\sup_{s\in\aleph_t}|r_n(s)| = O_p(n^{-1}\log n)$. Note that Lo *et al.* (1989) assume that the lifetimes are non-negative, whereas we work with random variables defined on the whole real line. However, it can be easily seen that their resuls remain valid in our setup. Then,

$$A_{11,n} = \frac{1}{nh}\sum_{i=1}^{n}U_{in}(t) + O_p((nh)^{-1}\log n),$$

where $U_{in}(t) = \int_{-1}^{1}\xi_i(t-uh)K'(u)du$. Therefore,

$$A_{1,n} = \frac{1}{nh}\sum_{i=1}^{n}U_{in}(t) + O_p(n^{-1/2}), \tag{7.9}$$

thanks to assumption (A.2).

Finally, we consider the term $A_{2,n}$. Under assumptions 3.1 and 3.2, we have

$$\begin{aligned} |A_{2,n}| &\leq f_*(t)\sup_{s\in\aleph_t}|\widehat{F}(s)-F(s)|\sup_{s\in\aleph_t}|f_*'(s)|(\inf_{s\in\aleph_t} f_*(s))^{-2} \\ &= O_p(n^{-1/2}). \end{aligned}$$

Therefore,

$$\widehat{f}_*(t) - \tilde{f}(t) = \frac{1}{nh} \sum_i U_{in}(t) + O_p(n^{-1/2}). \tag{7.10}$$

The result now follows from expressions (7.8) and (7.10).

2. We have

$$\begin{aligned}
\widetilde{f}(t) - f(t) &= \frac{f_*(t)}{h} \int_{t-h}^{t+h} K\left(\frac{t-s}{h}\right) \frac{f(s)}{f_*(s)} ds - f(t) \\
&= f_*(t) \int_{-1}^{1} K(u) r_*(t-uh) du - f(t) \\
&= f_*(t) \int_{-1}^{1} K(u) \big(r_*(t) - r_*'(t) uh + \frac{1}{2} h^2 r_*''(t) u^2 + o(h^2) \big) du - f(t) \\
&= \frac{1}{2} h^2 r_*''(t) f_*(t) \mu_K^2 + o(h^2).
\end{aligned}$$

Now, the result is an immediate consequence of the first point, and of Theorem 3.2 and Corollary 3.3 in Lo *et al.* (1989). □

Proof of Theorem 3.2. Write

$$(nh)^{1/2}(\widehat{f}_{\widehat{\theta}}(t) - f(t)) = (nh)^{1/2}(\widehat{f}_{\widehat{\theta}}(t) - \widehat{f}_{\theta*}(t)) + (nh)^{1/2}(\widehat{f}_{\theta^*}(t) - f(t)),$$

where $\widehat{f}_{\theta^*}(t)$ is the PGK density estimator based on the parametric guide $f_{\theta^*}(\cdot)$. From the second point in Theorem 3.1 it follows that

$$\sqrt{nh}\left(\widehat{f}_{\theta^*}(t) - f(t) - \frac{1}{2} h^2 \mu_K^2 r_{\theta^*}''(t) f_{\theta^*}(t) + o(h^2) \right) \xrightarrow{d} \mathcal{N}\left(0, \sigma^2(t)\right). \tag{7.11}$$

On the other hand, we have

$$\begin{aligned}
\widehat{f}_{\widehat{\theta}}(t) - \widehat{f}_{\theta^*}(t) &= \frac{1}{h}(f_{\widehat{\theta}}(t) - f_{\theta^*}(t)) \int_{t-h}^{t+h} K\left(\frac{t-s}{h}\right) \frac{1}{f_{\widehat{\theta}}(s)} d\widehat{F}(s) \\
&\quad - \frac{f_{\theta^*}(t)}{h} \int_{t-h}^{t+h} K\left(\frac{t-s}{h}\right) \frac{f_{\widehat{\theta}}(s) - f_{\theta^*}(s)}{f_{\widehat{\theta}}(s) f_{\theta^*}(s)} d\widehat{F}(s) \\
&= I_{1n} - I_{2n}.
\end{aligned}$$

By a Taylor expression we have, for an intermediate point θ_m between $\widehat{\theta}$ and θ^* and a constant $C < \infty$,

$$\begin{aligned} |I_{1n}| &\leq \frac{C}{h}\big\|\nabla_\theta f_{\theta_m}(t)\big\|\,|\widehat{\theta}-\theta^*|\,|\widehat{F}(t+h)-\widehat{F}(t-h)| \\ &\leq \frac{C}{h}\big\|\nabla_\theta f_{\theta_m}(t)\big\|\,|\widehat{\theta}-\theta^*|\,|\widehat{F}(t+h)-\widehat{F}(t-h)-F(t+h)+F(t-h)| \\ &\quad +\frac{C}{h}\big\|\nabla_\theta f_{\theta_m}(t)\big\|\,|\widehat{\theta}-\theta^*|\,|F(t+h)-F(t-h)|, \end{aligned}$$

where $\nabla_\theta f_\theta(t) = (\partial f_\theta(t)/\partial\theta_j)_{j=1}^p$. From Lemma 3 in Gijbels and Veraverbeke (1989), we have

$$\widehat{F}(t+h)-\widehat{F}(t-h)-F(t+h)+F(t-h) \;=\; O_p(n^{-1/2}h^{1/2}(\log n)^{1/2}).$$

Hence, $I_{1n} = o_p((nh)^{-1/2})$. In similar way it can be shown that $I_{2n} = o_p((nh)^{-1/2})$, and so

$$\widehat{f}_{\widehat{\theta}}(t) - \widehat{f}_{\theta^*}(t) = o_p((nh)^{-1/2}). \tag{7.12}$$

The result of Theorem 3.2 now follows from equations (7.11) and (7.12). □

Proof of Theorem 3.3. We have

$$(nh)^{1/2}(\widehat{\lambda}_{\widehat{\theta}}(t)-\lambda(t))) = (nh)^{1/2}\widehat{f}_{\widehat{\theta}}(t)\left[\frac{\widehat{F}(t)-F(t)}{(1-\widehat{F}(t))(1-F(t))}\right] + (nh)^{1/2}\left[\frac{\widehat{f}_{\widehat{\theta}}(t)-f(t)}{1-F(t)}\right].$$

Since the first term on the right hand side converges to zero in probability, the result of Theorem 3.3 is a direct consequence of Theorem 3.2. □

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Table 1: Squared bias ($\times 10^4$), Variance ($\times 10^4$), MSE ($\times 10^4$) and the optimal bandwidth h of the estimators of several normal mixture densities, for samples of size $n = 400$, with a censoring rate of 50% and $N = 1000$ replications.

	Method	Bias^2	Var	MSE	h
#1	PGK	0.015	4.376	4.391	0.332
	TK	3.764	8.556	12.32	0.736
#4	PGK	222.6	305.6	528.2	0.050
	TK	229.4	304.9	534.3	0.050
#6	PGK	0.003	2.352	2.355	3.234
	TK	0.008	0.350	0.358	1.582
#7	PGK	0.207	0.866	1.073	0.171
	TK	0.196	0.860	1.056	0.171
#9	PGK	0.001	2.663	2.664	2.307
	TK	3.479	1.161	4.640	0.856
#10	PGK	18.72	135.3	154.0	0.050
	TK	19.40	135.1	154.5	0.050
#11	PGK	0.006	2.977	2.983	0.775
	TK	37.99	1.590	39.58	0.735
#13	PGK	0.001	1.563	1.564	3.476
	TK	0.012	0.305	0.317	1.783
#14	PGK	0.381	1.210	1.591	0.131
	TK	0.368	1.204	1.572	0.131

Table 2: Squared bias ($\times 10^6$), Variance ($\times 10^6$), MSE ($\times 10^6$) and the optimal bandwidth h for the estimators of several Weibull densities for $a = (1, 2, 4)$, two censoring rates $p = (10\%, 40\%)$, two sample sizes $n = (150, 400)$ and $N = 1000$ replications.

p			10%				40%			
n	a	Method	Bias^2	Var	MSE	h	Bias^2	Var	MSE	h
150	1	PGK	0.009	114.6	114.6	5	0.579	194.2	194.8	5
		TK	23.60	99.70	123.3	5	26.60	153.7	180.3	5
	2	PGK	99.60	251.1	350.7	3.367	123.7	334.2	457.9	3.713
		TK	114.5	274.1	388.6	3.020	152.9	345.7	498.6	3.268
	4	PGK	98.91	258.1	357.0	3.169	94.40	377.0	471.4	3.268
		TK	246.6	514.4	761.0	2.228	303.7	629.2	932.9	2.327
n	a	Method	Bias^2	Var	MSE	h	Bias^2	Var	MSE	h
400	1	PGK	0.020	40.58	40.60	5	0.01	68.66	68.67	5
		TK	20.63	39.89	60.52	4.802	22.35	58.59	80.94	5
	2	PGK	50.70	124.8	175.5	2.872	62.80	177.7	240.5	3.020
		TK	54.20	146.0	200.2	2.525	75.00	189.3	264.3	2.723
	4	PGK	44.20	136.3	180.5	2.822	51.80	187.6	239.4	2.921
		TK	117.5	293.8	411.3	1.882	141.8	330.5	472.3	1.981

Table 3: Squared bias ($\times 10^6$), Variance ($\times 10^6$), MSE ($\times 10^6$) and the optimal bandwidth h for the estimators of several Weibull hazards for $a = (1, 2, 4)$, two censoring rates $p = (10\%, 40\%)$, two sample sizes $n = (150, 400)$ and $N = 1000$ replications.

p				10%				40%		
n	a	Method	Bias^2	Var	MSE	h	Bias^2	Var	MSE	h
150	1	PGK	0.360	90.77	91.13	5	1.300	145.0	146.3	5
		TK	14.69	113.0	127.7	4.901	18.60	168.6	187.2	5
	2	PGK	37.10	128.0	165.1	3.664	43.90	167.1	211.0	4.010
		TK	35.40	166.4	201.8	3.168	52.40	206.9	259.3	3.515
	4	PGK	24.20	91.4	115.6	3.366	24.70	119.6	144.2	3.465
		TK	48.00	169.20	218.2	2.327	71.90	189.4	261.3	2.525
n	a	Method	Bias^2	Var	MSE	h	Bias^2	Var	MSE	h
400	1	PGK	0.190	34.91	35.10	5	0.220	57.13	57.35	5
		TK	8.200	47.40	55.60	4.307	13.30	68.50	81.80	4.901
	2	PGK	17.20	63.50	80.70	3.020	21.80	85.10	106.9	3.218
		TK	15.80	80.90	96.70	2.574	22.90	102.3	125.12	2.822
	4	PGK	10.36	46.39	56.75	2.971	12.06	56.66	68.72	3.070
		TK	24.50	87.10	111.6	1.981	26.40	98.7	125.1	2.030