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Tests for the equality of conditional variance functions in nonparametric regression

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Abstract

In this paper we are interested in checking whether the conditional variances are equal in $k \geq 2$ location-scale models. Our procedure is fully nonparametric and is based on the comparison of the error distributions under the null hypothesis of equality of variances and without making use of this null hypothesis. We propose four test statistic based on empirical distribution functions (Kolmogorov-Smirnovand and Cramér-von Mises type test statistics) and two test statistics based on empirical characteristic functions. The limiting distributions of these six test statistics are established under the null hypothesis and under a local alternative. We show how to approximate the critical values under null using either an estimated version of the asymptotic null distribution or a bootstrap procedure. Simulation studies are conducted to assess the finite sample performance of the proposed tests. We also apply our tests to data on monthly expenditure.

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Key Words: Asymptotic analysis; bootstrap; empirical characteristic function; empirical distribution function; kernel smoothing; local alternatives; residuals.

1 Introduction

When comparing k ($k \geq 2$) populations it is interesting not only comparing the means, but also other characteristics like the variances. For example, in quality control, it is important to check the uniformity and the stability of the production process under different experimental and practical conditions. In biomedical research, detecting variation in gene expression levels is important for many reasons, for example, to identify experimental and environmental factors that affect a biological process; for a concrete example, see e.g. Mathur and Dolo (2008). Equality of variances, when it verified, can also be used to develop more powerful and simple ANOVA-type test statistics. Without controlling for the effect of covariates, there are a substantial number of tests available in the literature for the equality of (unconditional) variances from two or more populations. The standard procedures include the classical F-test and Levene's test (Levene, 1960) which is known to be more robust to the violation of normality; see Gastwirth et al. (2009) for a recent review and some interesting examples and applications. In this paper, we are interested in the comparison of conditional variances.

We assume that in each population, along with the variable of interest or response variable, Y, it is also observed another variable, X, the covariate, so that the mean and the variance of the response variable depend on the values of X. More specifically, let (X_j, Y_j), $1 \leq j \leq k$, be k independent random vectors satisfying general nonparametric regression models

$$Y_j = m_j(X_j) + \sigma_j(X_j)\varepsilon_j, \tag{1}$$

where $m_j(x) = E(Y_j \mid X_j = x)$ is the regression function, $\sigma_j^2(x) = Var(Y_j \mid X_j = x)$ is the conditional variance function and ε_j is the regression error, which is assumed to be independent of X_j. Note that, by construction, $E(\varepsilon_j) = 0$ and $Var(\varepsilon_j)$=1. The covariate X_j is continuous with density function f_j. Since the objective is to compare the variance functions, it is reasonable to assume that the covariates have common support, say R. The regression functions, the variance functions, the distribution of the errors and the distribution of the covariates are completely unknown and no parametric models are assumed for them. Thus, our approach is completely nonparametric. In this conditional setting, the hypothesis of equality of variances is stated in terms of the conditional variance functions: $H_0 : \sigma_1^2(x) = \sigma_2^2(x) = \ldots = \sigma_k^2(x)$, $\forall x \in R$. Or equivalently,

$$H_0 : \sigma_j(x)/\sigma_0(x) = 1, \quad \text{for } j = 1\ldots,k,$$

where $\sigma_0^2(x)$ is a common variance that can be expressed us $\sigma_0^2(x) = \sum_{j=1}^k \pi_j(x)\sigma_j^2(x)$, for some positive functions $\pi_1,\ldots,\pi_k$ satisfying $\sum_{j=1}^k \pi_j(x) = 1$. The alternative hypothesis is

$$H_1 : \sigma_j(x)/\sigma_0(x) \neq 1, \quad \text{for some } j \in \{1\ldots,k\},$$

We will develop several test statistics and study their distribution under H_0 and under local alternatives converging to the null hypothesis at the rate $n^{-1/2}$; n being the sample size. Specifically, we consider the following local alternative hypothesis

$$H_{1,n} :\ \sigma_{n,j}(x)/\sigma_{n,0}(x) = 1 + n^{-1/2}\delta_j(x), \quad \text{for } j = 1\ldots,k,$$

for some continuous functions δ_j. In order to shorten the notation, we will suppress the explicit dependence on n and simply write $\sigma_j(x)/\sigma_0(x) = 1 + n^{-1/2}\delta_j(x)$. Observe that as n increases $H_{1,n}$ becomes closer and closer to H_0. Also, when $\delta_j(x) = 0$, for $j = 1,\ldots,k$, $H_{1,n}$ reduces to H_0.

Statistical literature concerning the problem of testing for common features in several regression models has mainly focused on testing for common regression curves or testing for common error distribution. The problem of testing for the equality of regression curves in nonparametric settings has been extensively treated; see for example Delgado (1993), Kulasekera (1995), Neumeyer and Dette (2003), Pardo-Fernández *et al.* (2007, 2012), Srihera and Stute (2010) and González-Manteiga and Crujeiras (2013) for a recent review. On the other hand, testing for the equality of error distributions has been addressed in Pardo-Fernández (2007). To the best of our knowledge, the comparison of conditional variance functions has not been studied before. Most papers dealing with testing on the conditional variance function focus on studying the hypothesis of homoscedasticity, $\sigma(.) = \sigma > 0$, $\sigma \in \mathbb{R}$ (see for example Liero, 2003, or Dette and Marchlewski, 2010, and the references therein), or more in general, if the conditional variance function follows some fixed parametric form (see for example Dette *et al.*, 2007, or Koul and Song, 2010, and the references therein).

In order to construct a test for testing H_0, several approaches are possible. Here we follow the ideas in Pardo-Fernández *et al.* (2007, 2012) for testing the equality of the regression functions, $m_1,\ldots,m_k$, which consists of comparing the distribution of the errors of the regression models. Specifically, let

$$\varepsilon_j = \frac{Y_j - m_j(X_j)}{\sigma_j(X_j)}, \tag{2}$$

be the regression error in population j, $1 \le j \le k$. Define

$$\varepsilon_{0j} = \frac{Y_j - m_j(X_j)}{\sigma_0(X_j)} = \varepsilon_j \frac{\sigma_j(X_j)}{\sigma_0(X_j)} \tag{3}$$

to be the error under the null hypothesis, $1 \le j \le k$. Let $F_{\varepsilon_j}(t) = P(\varepsilon_j \le t)$ and $F_{\varepsilon_{0j}}(t) = P(\varepsilon_{0j} \le t)$ be the cumulative distribution function (CDF) of ε_j and ε_{0j}, respectively. The following Theorem justifies that H_0 is true if and only if the distributions of ε_j and ε_{0j} coincide. The proof is deferred to the Appendix.

Theorem 1 *Assume that σ_j is a continuous function in R and $0 < E(\varepsilon_j^4) < \infty$, $1 \le j \le k$.*

(a) H_0 is true if and only if the random variables ε_j and ε_{0j} have the same distribution for all $1 \le j \le k$.

(b) Let $p_1, \ldots, p_k$ be such that $p_j > 0$, $1 \le j \le k$, and $\sum_{j=1}^k p_k = 1$. Let $F_\varepsilon(t) = \sum_{j=1}^k p_j F_{\varepsilon_j}(t)$ and $F_{\varepsilon_0}(t) = \sum_{j=1}^k p_j F_{\varepsilon_{0j}}(t)$. Assume also that $E(\varepsilon_1^4) = \cdots = E(\varepsilon_k^4)$. Then H_0 is true if and only if $F_\varepsilon(t) = F_{\varepsilon_0}(t)$, for all $t \in \mathbb{R}$.

The assertions in the previous result can be interpreted in terms of the CDF or in terms of any other function characterizing a probability law, such as the characteristic function (CF). In this paper we will consider both cases, that is, to test H_0 we will compare consistent estimators of the CDFs and CFs of the random variables ε_j and ε_{0j}, $1 \le j \le k$.

With this aim, the paper is organized as follows. In the next section we introduce the test statistics and explain the testing procedure. Section 3 and 4 contains the main asymptotic results concerning the ECDF-based test statistics and the ECF-based test statistics, respectively, and discusses some practical considerations. In Section 5 we explain how the critical values of the proposed test statistics can be approximated. Investigating the finale sample performance of our tests is the topic of Section 6. A data example follows in Section 7 and conclusions are given in Section 8. All proofs of the theoretical results are deferred to the Appendix.

The following notation will be used along the paper: P_0 denotes probability assuming that H_0 is true; E_0 denotes expectation assuming that H_0 is true; P_* denotes the conditional probability law, given the data; all limits in this paper are taken when $n \to \infty$; $\stackrel{\mathcal{L}}{\longrightarrow}$ denotes convergence in distribution; $\stackrel{P}{\longrightarrow}$ denotes convergence in probability; if $x \in \mathbb{R}^k$, with $x' = (x_1, \ldots, x_k)$, then $diag(x)$ is the $k \times k$ diagonal matrix whose (i,i) entry is x_i, $1 \le i \le k$; for any complex number $z = a + \mathrm{i}b$, $Re(z) = a$ is its real part, $Im(z) = b$

is its real part, $\bar{z} = a - ib$ is its conjugate and $|z|$ is its modulus; $N_k(\mu, \Sigma)$ denotes the multivariate normal distribution with mean vector μ and variance-covariance matrix Σ; an unspecified integral denotes integration over the whole real line $\mathbb{R}$; $\sup_t$ stands for $\sup_{t\in\mathbb{R}}$.

2 The test statistics

As in the Introduction, let (X_j, Y_j), $1 \leq j \leq k$, be k independent random vectors satisfying general nonparametric regression models (1). For $1 \leq j \leq k$, let ε_j and ε_{0j} be as defined in (2) and (3), respectively. As justified in Theorem 1, to test for H_0 we will compare consistent estimators of the CDFs and CFs of the random variables ε_j and ε_{0j}, $1 \leq j \leq k$, and also consistent estimators of the CDFs F and F_0 and of their associated CFs. Since neither ε_j nor ε_{0j} are observable, the inference must be based on the estimated residuals. Next we construct them.

Let (X_{jl}, Y_{jl}), $1 \leq l \leq n_j$, be independent, identically distributed (iid) observations from (X_j, Y_j), $1 \leq j \leq k$. In order to estimate the errors, we first need to estimate the regression functions, $m_j(x) = E(Y_j|X_j = x)$, the variance functions, $\sigma_j^2(x) = E[\{Y_j - m_j(x)\}^2|X_j = x]$, and the common variance function under H_0, $\sigma_0^2(x)$. With this aim we use nonparametric estimators based on kernel smoothing techniques. Let K denote a nonnegative kernel function defined on $\mathbb{R}$, let $0 < h_n \equiv h \to 0$ be the bandwidth or smoothing parameter and $K_h(x) = h^{-1}K(x/h)$. We use the following estimators for the functions m_j, σ_j^2 and σ_0^2:

$$\hat{m}_j(x) = \sum_{l=1}^{n_j} w_{jl}(x) Y_{jl}, \quad \hat{\sigma}_j^2(x) = \sum_{l=1}^{n_j} w_{jl}(x) Y_{jl}^2 - \hat{m}_j^2(x), \quad \hat{\sigma}_0^2(x) = \sum_{j=1}^{k} \pi_j(x) \hat{\sigma}_j^2(x).$$

The quantities w_{jl} are, either the local-linear weights given by

$$w_{jl}(x) = \frac{K_h(X_{jl} - x)\left(S_{2,n_j}(x) - (X_{jl} - x)S_{1,n_j}\right)}{S_{0,n_j}(x)S_{2,n_j}(x) - S_{1,n_j}^2(x)},$$

with $S_{k,n_j}(x) = \sum_{l=1}^{n_j}(X_{jl} - x)^k K_h(X_{jl} - x)$, $k = 0, 1, 2$, or the Nadaraya-Watson weights

$$w_{jl}(x) = \frac{K_h(X_{jl} - x)}{\sum_{v=1}^{n_j} K_h(X_{jv} - x)}.$$

Both are particular cases of local-polynomial weighting (see Fan and Gijbles, 1996). Under the model assumptions that will be stated in the next section, the results in this article are valid for local-linear and for Nadaraya-Watson (local-constant) estimators. Note that we have implicitly assumed that the functions $\pi_1, \ldots, \pi_k$ do not depend on unknowns. The theory also apply to the case where they depend on unknowns, replacing π_j by $\hat{\pi}_j$ in the expression $\hat{\sigma}_0^2(x)$, whenever $\hat{\pi}_j$ converges to π_i fast enough. Later we will discuss this issue in more detail.

Based on these estimators, for each population j, $1 \le j \le k$, we construct two samples of residuals,

$$\hat{\varepsilon}_{jl} = \frac{Y_{jl} - \hat{m}_j(X_{jl})}{\hat{\sigma}_j(X_{jl})} \qquad \text{and} \qquad \hat{\varepsilon}_{0jl} = \frac{Y_{jl} - \hat{m}_j(X_{jl})}{\hat{\sigma}_0(X_{jl})}, \tag{4}$$

$1 \le l \le n_j$. Then we can construct the corresponding empirical CDFs (ECDFs),

$$\hat{F}_{\varepsilon_j}(t) = \frac{1}{n_j}\sum_{l=1}^{n_j} I(\hat{\varepsilon}_{jl} \le t) \qquad \text{and} \qquad \hat{F}_{\varepsilon_{0j}}(t) = \frac{1}{n_j}\sum_{l=1}^{n_j} I(\hat{\varepsilon}_{0jl} \le t),$$

and empirical CFs (ECFs),

$$\hat{\varphi}_{\varepsilon_j}(t) = \frac{1}{n_j}\sum_{l=1}^{n_j} \exp(\mathrm{i}t\hat{\varepsilon}_{jl}) \qquad \text{and} \qquad \hat{\varphi}_{\varepsilon_{0j}}(t) = \frac{1}{n_j}\sum_{l=1}^{n_j} \exp(\mathrm{i}t\hat{\varepsilon}_{0jl}),$$

respectively. These ECDFs are consistent kernel based nonparametric estimators of the population CDFs $F_{\varepsilon_j}(t)$ and $F_{\varepsilon_{0j}}(t)$, respectively (see Theorem 2 below). Analogously, the above ECFs are consistent kernel based nonparametric estimators of the population CFs $\varphi_{\varepsilon_j}(t) = E\{\exp(\mathrm{i}t\varepsilon_j)\}$ and $\varphi_{\varepsilon_{0j}}(t) = E\{\exp(\mathrm{i}t\varepsilon_{0j})\}$, respectively (see Theorem 6 below). We can also consider the following ECDFs

$$\hat{F}_{\varepsilon}(t) = \frac{1}{n}\sum_{j=1}^{k}\sum_{l=1}^{n_j} I(\hat{\varepsilon}_{jl} \le t) \qquad \text{and} \qquad \hat{F}_{\varepsilon_0}(t) = \frac{1}{n}\sum_{j=1}^{k}\sum_{l=1}^{n_j} I(\hat{\varepsilon}_{0jl} \le t),$$

and ECFs,

$$\hat{\varphi}_{\varepsilon}(t) = \frac{1}{n}\sum_{j=1}^{k}\sum_{l=1}^{n_j} \exp(\mathrm{i}t\hat{\varepsilon}_{jl}) \qquad \text{and} \qquad \hat{\varphi}_{\varepsilon_0}(t) = \frac{1}{n}\sum_{j=1}^{k}\sum_{l=1}^{n_j} \exp(\mathrm{i}t\hat{\varepsilon}_{0jl}),$$

that estimate $F_{\varepsilon}(t) = \sum_{j=1}^k p_j F_{\varepsilon_j}(t)$, $F_{\varepsilon_0}(t) = \sum_{j=1}^k p_j F_{\varepsilon_{0j}}(t)$, $\varphi_{\varepsilon}(t) = \sum_{j=1}^k p_j \varphi_{\varepsilon_j}(t)$ and $\varphi_{\varepsilon_0}(t) = \sum_{j=1}^k p_j \varphi_{\varepsilon_{0j}}(t)$, respectively, where $n = \sum_{j=1}^k n_j$ and $n_j/n \to p_j$, $1 \le j \le k$.

To test for H_0, we will construct Kolmogorov-Smirnov type statistics and Cramér-von Mises type statistics to compare the ECDFs, and weighted L_2-distances to compare the ECFs. More precisely, the considered statistics are:

$$T_{KS}^1 = \sum_{j=1}^{k} \sqrt{n_j} \sup_t |\hat{F}_{\varepsilon_j}(t) - \hat{F}_{\varepsilon_{0j}}(t)|, \quad T_{CM}^1 = \sum_{j=1}^{k} n_j \int \{\hat{F}_{\varepsilon_j}(t) - \hat{F}_{\varepsilon_{0j}}(t)\}^2 d\hat{F}_{\varepsilon_{0j}}(t)$$

$$T_{KS}^2 = \sqrt{n} \sup_t |\hat{F}_\varepsilon(t) - \hat{F}_{\varepsilon_0}(t)|, \quad T_{CM}^2 = n \int \{\hat{F}_\varepsilon(t) - \hat{F}_{\varepsilon_0}(t)\}^2 d\hat{F}_0(t),$$

$$T_1 = \sum_{j=1}^{k} n_j \int \left|\hat{\varphi}_{\varepsilon_j}(t) - \hat{\varphi}_{\varepsilon_{0j}}(t)\right|^2 w(t)dt, \quad T_2 = n \int |\hat{\varphi}_\varepsilon(t) - \hat{\varphi}_{\varepsilon_0}(t)|^2 w(t)dt,$$

where w is a positive weight function that is needed to guarantee consistency; see Section 4. Note that in the case of T_1 and T_2, $|\cdot|$ represents the modulus of a complex number. In Section 3 we will study studies the asymptotic properties of the statistics T_{KS}^1, T_{CM}^1, T_{KS}^2 and T_{CM}^2 and in Section 4 we will deal with T_1 and T_2.

3 Asymptotics for ECDF-based test statistics

This section studies some asymptotic properties of the ECDF-based test statistics T_{KS}^1, T_{CM}^1, T_{KS}^2 and T_{CM}^2. To derive such properties we will need some commonly assumed regularity assumptions. First let define $F_j(t|x) = P(Y_j \leq t|X_j = x)$ and $F_j(x) = P(X_j \leq x)$, for $1 \leq j \leq k$.

Assumption (A1): For $1 \leq j \leq k$,

- **(i)** X_j is absolutely continuous with compact support R and density f_j,
- **(ii)** f_j, m_j, σ_j and π_j are twice continuously differentiable on R,
- **(iii)** $\inf_{x \in R} f_j(x) \geq c > 0$ and $\inf_{x \in R} \sigma_j(x) \geq d > 0$, for some $c, d \in \mathbb{R}$,
- **(iv)** $E(\varepsilon_j^4) < \infty$.
- **(vi)** $nh_n^4 \to 0$ and $nh_n^{3+2\delta}(\log h_n^{-1})^{-1} \to \infty$, for some $\delta > 0$.
- **(vii)** The kernel K is a symmetric density function with compact support and it is twice continuously differentiable.

Assumption (A2): For $1 \leq j \leq k$, $F_j(t|x)$ is continuous in (x,t) and differentiable with respect to t, $\frac{\partial}{\partial t} F_j(t|x) = F_j'(t|x)$ is continuous in (x,t) and $\sup_{x,t} |t^2 F_j'(t|x)| < \infty$. The same holds for all other partial derivatives of $F_j(t|x)$ with respect to x and t up to order two.

From now on we will name Assumption A to be the set of Assumptions (A1)–(A2). Assumption A (skipping (A1)(iv)) was also considered in Pardo-Fernández *et al.* (2007) to derive asymptotic properties of some ECDF-based tests designed to detect differences between the conditional mean functions. This assumption is mainly needed to guarantee the uniform consistency of the estimators $\hat{f}_j$, $\hat{\sigma}_j$, $\hat{m}_j$ and $\hat{\sigma}_0$.

We first give the following result that justifies the use of the test statistics T^1_{KS}, T^1_{CM}, T^2_{KS} and T^2_{CM} for testing H_0.

Theorem 2 *Suppose that Assumption A holds. Then,*

$$\hat{F}_{\varepsilon_{0j}}(t) = F_{\varepsilon_{0j}}(t) + o_p(1), \ \ \textit{and} \quad \hat{F}_{\varepsilon_j}(t) = F_{\varepsilon_j}(t) + o_p(1),$$

uniformly in t, $1 \le j \le k$.

Corollary 3 *Suppose that Assumption A holds. Then,*

$$\frac{1}{\sqrt{n}} T^1_{KS} \xrightarrow{P} \sum_{j=1}^{k} \sqrt{p_j} \sup_t |F_{\varepsilon_j}(t) - F_{\varepsilon_{0j}}(t)|, \frac{1}{n} T^1_{CM} \xrightarrow{P} \sum_{j=1}^{k} p_j \int \{F_{\varepsilon_j}(t) - F_{\varepsilon_{0j}}(t)\}^2 dF_{\varepsilon_{0j}}(t),$$
$$\frac{1}{\sqrt{n}} T^2_{KS} \xrightarrow{P} \sup_t |F_{\varepsilon}(t) - F_{\varepsilon_0}(t)|, \ \textit{and} \ \frac{1}{n} T^2_{CM} \xrightarrow{P} \int \{F_{\varepsilon}(t) - F_{\varepsilon_0}(t)\}^2 dF_0(t).$$

Observe that all considered test statistics converge in probability to non-negative quantities. Under the assumptions in Theorem 1, such quantities are 0 if and only if H_0 is true. Therefore it seems reasonable to reject the null hypothesis for *large* values of the considered test statistics. Now, to determine what large values mean in each case, we must calculate the null distribution of the test statistic, or at least an approximation to it. Since the null distributions are unknown, we derive their asymptotic null distributions.

Theorem 4 *Suppose that Assumption A holds. Then, under $H_{1,n}$,*

$$\sqrt{n_j}(\hat{F}_{\varepsilon_j}(t) - \hat{F}_{\varepsilon_{0j}}(t)) = \frac{1}{2} t f_{\varepsilon_j}(t)(p_j^{1/2}\Delta_j + Z_{n,j}) + o_p(1),$$

uniformly in t, where $\Delta_j = 2E(\delta_j(X_j))$, and

$$Z_{n,j} = \sqrt{n_j} \sum_{v=1}^{k} \frac{1}{n_v} \sum_{l=1}^{n_v} \left\{ I(v=j) - \pi_v(X_{vl}) \frac{f_j(X_{vl})}{f_v(X_{vl})} \right\} \left(\varepsilon_{vl}^2 - 1 \right). \tag{5}$$

The following Corollary, derived mainly by applying the multivariate CLT to $Z_n = (Z_{n,1}, \ldots, Z_{n,k})'$, gives the asymptotic distribution of our ECDF-based test statistics under the H_0 and $H_{1,n}$.

Corollary 5 *Suppose that Assumption A holds. Then, under $H_{1,n}$,*

$$T^1_{KS} \xrightarrow{\mathcal{L}} \frac{1}{2} \sum_{j=1}^{k} |Z_j + p_j^{1/2}\Delta_j| \sup_t |t f_{\varepsilon_j}(t)|, \quad T^1_{CM} \xrightarrow{\mathcal{L}} \frac{1}{4} \sum_{j=1}^{k} (Z_j + p_j^{1/2}\Delta_j)^2 \int t^2 f^2_{\varepsilon_j}(t) dF_{\varepsilon_j}(t),$$
$$T^2_{KS} \xrightarrow{\mathcal{L}} \frac{1}{2} \sup_y |Z(t) + \Delta(t)|, \qquad T^2_{CM} \xrightarrow{\mathcal{L}} \frac{1}{4} \int \{Z(t) + \Delta(t)\}^2 dF_{\varepsilon}(t),$$

where $Z(t)=\sum_{j=1}^k p_j^{1/2} t f_{\varepsilon_j}(t) Z_j$*,* $\Delta(t)=\sum_{j=1}^k p_j t f_{\varepsilon_j}(t)\Delta_j$*, and* $(Z_1,\ldots,Z_k)'\sim N_k(0,\Sigma)$*, with* $\Sigma=(\sigma_{jv})$ *being the* $k\times k$*-matrix whose elements are*

$$\sigma_{jv}=(p_j p_v)^{1/2}\sum_{l=1}^k \frac{E\{(\varepsilon_l^2-1)^2\}}{p_l} E\left[\left\{\pi_l(X_l)\frac{f_j(X_l)}{f_l(X_l)}-I(l=j)\right\}\left\{\pi_l(X_l)\frac{f_v(X_l)}{f_l(X_l)}-I(l=v)\right\}\right] \quad (6)$$

$1\le j,v\le k$*, where* $I(S)$ *denotes the indicator function of a set* S*.*

Let T denote any of the test statistics T^1_{KS}, T^1_{CM}, T^2_{KS} and T^2_{CM}. Since H_0 can be seen us a special case of $H_{1,n}$ with $\delta_j=0, j=1,\ldots,k$, the asymptotic distribution of T under the null follows trivially by setting $\Delta_j=0$. That is to say that, for example, under H_0,

$$T^1_{KS}\xrightarrow{\mathcal{L}}\frac{1}{2}\sum_{j=1}^k |Z_j|\sup_t |t f_{\varepsilon_j}(t)|, \text{ and } T^2_{KS}\xrightarrow{\mathcal{L}}\frac{1}{2}\sup_t |Z(t)|.$$

Let $\alpha\in(0,1)$ be arbitrary but fixed. As an immediate consequence of Theorem 1 and Corollaries 3 and 5, the test that rejects H_0 when $T\ge t_\alpha$, where t_α is the $1-\alpha$ percentile of the null distribution of T or any consistent estimator of it, is consistent against all fixed alternatives. It is also able to detect local alternatives converging to the null at the rate $n^{1/2}$, whenever $\Delta_j\ne 0$ for some $1\le j\le k$.

So far we have assumed that the weight functions $\pi_1,\ldots,\pi_k$ are known. Nevertheless we didn't make any restriction on them except the fact that they are positive and sum to one. In our simulation study, see Section 6, we tack $\pi_j=n_j/n$. This simple choice is shown to work reasonably well for all the investigated examples. However, it may be more appropriate to select the π_j's from the data. For example, as for the problem of testing the equality of regression curves, see Pardo-Fernández *et al.* (2007, 2012), one my choose $\pi_j(x)=p_j f_j(x)/f_{mix}(x)$, with $f_{mix}(x)=\sum_{j=1}^k p_j f_j(x)$. For this choice, since $f_1,\ldots f_k$ are unknown, the functions $\pi_1,\ldots,\pi_k$ must be estimated. A careful reading of the proofs reveals that all the results in this paper continue to be true whenever $\pi_1,\ldots,\pi_k$ are replaced by convenient estimators $\hat\pi_1,\ldots,\hat\pi_k$ satisfying

$$\sup_{x\in R}|\pi_j(x)-\hat\pi_j(x)|=o_p(n^{-1/4}),\quad 1\le j\le k. \quad (7)$$

4 Asymptotics for ECF-based test statistics

In order to study the limit behaviour of the test statistics T_1 and T_2 we also need some regularity conditions. Recall that to derive the asymptotic properties for the ECDF-based test statistics we assumed that the regression errors have a twice differentiable CDF.

Analogously, to derive the asymptotic properties for the ECF-based test statistics we need that the regression errors has a twice differentiable CF, which is tantamount to assume that the regression errors has finite second order moment. But this assumption is implicit in the the definition of the regression models (1). As a consequence, the assumptions required to derive the asymptotics for ECF-based test statistics will be weaker than those assumed in Section 3, in the sense that no restriction on the distribution of the errors will be imposed, such as the existence of a density. Specifically, we mainly need to assume that Assumption (A1) holds. The motivation behind the test statistics T_1 and T_2 is in the following result.

Theorem 6 *Suppose that Assumption (A1) holds and that $w \geq 0$ is such that $\int t^2 w(t)dt < \infty$. Then, $n^{-1}T_i = \tau_i + o_p(1)$, $i = 1, 2$, where*

$$\tau_1 = \sum_{j=1}^{k} p_j \int \left|\varphi_{\varepsilon_j}(t) - \varphi_{\varepsilon_{0j}}(t)\right|^2 w(t)dt, \qquad \tau_2 = \int \left|\varphi_{\varepsilon}(t) - \varphi_{\varepsilon_0}(t)\right|^2 w(t)dt.$$

T_1 and T_2 converge in probability to non-negative quantities. Since two distinct CFs can be equal in a finite interval (see, for example, Feller, 1971; p. 479), a general way to ensure that $\tau_1 > 0$ and $\tau_2 > 0$ whenever $\sigma_r \neq \sigma_s$, for some $1 \leq r, s \leq k$, $r \neq s$, is to take $w(t) > 0$, for all $t \in \mathbb{R}$. For instance, one can take w as the pdf of a normal law. Now, the reasoning made just after Corollary 3 can be repeated for the test statistics T_1 and T_2. So our next goal is to determine the asymptotic distribution of T_1 and T_2. With this aim we first give a result that provides an asymptotic approximation for $\sqrt{n_j}\{\hat{\varphi}_{\varepsilon_j}(t) - \hat{\varphi}_{\varepsilon_{0j}}(t)\}$, $1 \leq j \leq k$. Let $\varphi'_{\varepsilon_j}(t) = \frac{\partial}{\partial t}\varphi_{\varepsilon_j}(t) = \frac{\partial}{\partial t}Re\varphi_{\varepsilon_j}(t) + \mathrm{i}\frac{\partial}{\partial t}Im\varphi_{\varepsilon_j}(t) = \mathrm{i}E(\varepsilon_j \exp(\mathrm{i}t\varepsilon_j))$, which exists because $E(|\varepsilon_j|) < \infty$, $1 \leq j \leq k$.

Theorem 7 *Suppose that Assumptions (A1) holds. Then, under $H_{1,n}$,*

$$\sqrt{n_j}(\hat{\varphi}_{\varepsilon_j}(t) - \hat{\varphi}_{\varepsilon_{0j}}(t)) = \frac{1}{2}t\varphi'_{\varepsilon_j}(t)(p_j^{1/2}\Delta_j - Z_{n,j}) + tR_{1j}(t) + t^2R_{2j}(t),$$

where $\sup_t |R_{sj}(t)| = o_p(1)$, $s = 1, 2$, and $Z_{n,j}$ and Δ_j, $j = 1, \ldots, k$ are defined as in Theorem 4.

Corollary 8 *Suppose that Assumptions (A1) holds and that $w \geq 0$ is such that $\int t^4 w(t)dt < \infty$. Then, under $H_{1,n}$,*

$$T_1 \xrightarrow{\mathcal{L}} \frac{1}{4}\sum_{j=1}^{k}(Z_j + p_j^{1/2}\Delta_j)^2 \int t^2|\varphi'_{\varepsilon_j}(t)|^2 w(t)dt, \qquad T_2 \xrightarrow{\mathcal{L}} \frac{1}{4}\int |V(t) + W(t)|^2 w(t)dt,$$

where $V(t) = \sum_{j=1}^{k} p_j^{1/2} t\varphi'_{\varepsilon_j}(t)Z_j$, $W(t) = \sum_{j=1}^{k} p_j t\varphi'_{\varepsilon_j}(t)\Delta_j$, and $(Z_1, \ldots, Z_k)' \sim N_k(0, \Sigma)$.

Similar comments to those made after Corollary 5 for the test statistics T^1_{KS}, T^1_{CM}, T^2_{KS} and T^2_{CM} can be done for T_1 and T_2.

Before ending this section, we give a brief discussion on the choice the weight function w. It has been seen that taking $w > 0$ ensures that the test rejecting H_0 for large values of T_1 or T_2, is consistent against any fixed alternative. It also ensures that T_1 converges in law, under H_0, to a non-degenerate distribution (see Section 5). From a theoretical point of view, any positive function w satisfying $\int t^4 w(t)dt < \infty$ can be used. From a practical point of view, the ease of computation of T_1 and T_2 is closely related to the choice of w. In fact, an alternative and more useful expression for T_1 and T_2 is given by (see Lemma 1 in Alba-Fernández et al., 2008)

$$T_1 = \sum_{j=1}^{k} \frac{1}{n_j} \left\{ \sum_{l,s=1}^{n_j} I_w(\hat{\varepsilon}_{jl} - \hat{\varepsilon}_{js}) + \sum_{l,s=1}^{n_j} I_w(\hat{\varepsilon}_{0jl} - \hat{\varepsilon}_{0js}) - 2 \sum_{l,s=1}^{n_j} I_w(\hat{\varepsilon}_{jl} - \hat{\varepsilon}_{0js}) \right\},$$

$$T_2 = \frac{1}{n} \sum_{j,v=1}^{k} \sum_{l=1}^{n_j} \sum_{s=1}^{n_v} \left\{ I_w(\hat{\varepsilon}_{jl} - \hat{\varepsilon}_{vs}) + I_w(\hat{\varepsilon}_{0jl} - \hat{\varepsilon}_{0vs}) - 2 I_w(\hat{\varepsilon}_{jl} - \hat{\varepsilon}_{0vs}) \right\},$$

where

$$I_w(t) = \int \cos(tx) w(x) dx. \qquad (8)$$

This expression is specially appealing when one wishes to employ the bootstrap to approximate the null distribution, which requires to evaluate the test statistic in a high number of artificial samples. Another point that should be taken into account is the fact that the ECF estimates more accurately the population CF around $t = 0$. Consequently, w should put most of the weight near the origin. For the problem of testing the equality of mean regression curves, Pardo-Fernández et al (2012) tack w to be the standard normal density. We also considered this choice for w in our simulation study.

5 Estimation of the null distribution

The results in Corollaries 5 and 8 reveal that the asymptotic null distributions of the proposed test statistics are in all cases unknown because they depend on unknown quantities. Therefore, the asymptotic null distribution cannot be directly used to approximate the null distribution of these statistics. Two solutions can be considered: (a) approximate the null distribution by a bootstrap procedure, or (b) construct an approximation of the asymptotic null distribution. The first approach was also considered in Pardo-Fernández *et al.* (2007) for the problem of testing the equality of conditional mean functions. They employed a bootstrap procedure based on smoothed residuals, whose theoretical justification can be found in Neumeyer (2009). The same bootstrap procedure could be used to approximate the null distribution of the test statistics studied in this paper.

The second possibility is to approximate the null distribution by means of an estimator of the asymptotic null distribution of the test statistics. This estimator is usually called a bootstrap in the limit estimator. Let us first consider the test statistic T^1_{CM}. According to Corollary 5, under H_0, $4T^1_{CM} \xrightarrow{\mathcal{L}} W_1 := \sum_{j=1}^k \alpha_j \chi^2_{1,j}$, where $\chi^2_{1,1}, \ldots, \chi^2_{1,k}$ are independent chi-square random variates with one degree of freedom and $\alpha_1, \ldots, \alpha_k$ are the eigenvalues of $\mathcal{A}\Sigma$, $\mathcal{A} = diag(a_1, \ldots, a_k)$, $a_j = \int t^2 f^2_{\varepsilon_j}(t) dF_{\varepsilon_j}(t)$, $1 \le j \le k$. Before employing a bootstrap in the limit estimator we must be sure that the asymptotic null distribution is non-degenerate. Since, under our assumptions, $a_j > 0$ and $\sigma_{jj} > 0$, $1 \le j \le k$, we have that $\sum_{j=1}^k \alpha_j = \text{trace}(\mathcal{A}\Sigma) = \sum_{j=1}^k a_j \sigma_{jj} > 0$, and therefore its asymptotic null distribution is non-degenerate. The quantities α_j in W_1 are unknown but can be estimated consistently from the data, say by $\hat{\alpha}_j$, the eigenvalues of $\hat{\mathcal{A}}\hat{\Sigma}$, using plug-in principle and kernel smoothing method. In such a case,

$$\sup_t \left| P_0\{T^1_{CM} \le t\} - P_*(\hat{W}_1 \le t) \right| \xrightarrow{P} 0,$$

where $\hat{W}_1 = \sum_{j=1}^k \hat{\alpha}_j \chi^2_{1j}$. Analogously, one could also estimate the null distribution of T^2_{CM}, T_1 and T_2.

As for T_1, Corollary 8 says that $4T_1$ converges in law to $W_2 = \sum_{j=1}^k \beta_j \chi^2_{1,j}$, where $\chi^2_{1,1}, \ldots, \chi^2_{1,k}$ are as before and $\beta_1, \ldots, \beta_k$ are the eigenvalues of $\mathcal{B}\Sigma$, $\mathcal{B} = diag(b_1, \ldots, b_k)$, with $b_j = \int t^2 |\varphi'_{\varepsilon_j}(t)|^2 w(t) dt$, $1 \le j \le k$. Since $\sigma_{jj} > 0$, $1 \le j \le k$, and $\sum_{j=1}^k \beta_j = \sum_{j=1}^k b_j \sigma_{jj}$, to ensure that W_2 is non-degenerate we must have that $b_j > 0$ for some $1 \le j \le k$. Since $E(\varepsilon_j) = \varphi'_{\varepsilon_j}(0) = 0$ and $E(\varepsilon_j^2) = -\varphi''_{\varepsilon_j}(0) = 1$, where $\varphi''_{\varepsilon_j}(t) = \frac{\partial^2}{\partial t^2} Re\varphi_{\varepsilon_j}(t) + \mathrm{i}\frac{\partial^2}{\partial t^2} Im\varphi_{\varepsilon_j}(t)$, it readily follows that $|\varphi'_{\varepsilon_j}(t)| > 0$, for all $t \in (-\delta, 0) \cup (0, \delta)$, for some $\delta > 0$. Thus, if the weight function w is positive in an open neighborhood of the origin, we have that $b_j > 0$, $1 \le j \le k$, which implies that W_2 is non-degenerate.

Interestingly, if all the covariates have the same distribution, $f_1 = \cdots = f_k$, $E[(\varepsilon_1^2 - 1)^2] = \cdots = E[(\varepsilon_k^2 - 1)^2] := \theta$ and $\pi_j(x) = p_j$, $1 \le j \le k$, then

$$\Sigma = \theta(I_k - pp'), \quad p' = (\sqrt{p_1}, \ldots, \sqrt{p_k}). \tag{9}$$

It is easy to see that the matrix $I_k - pp'$ has two different eigenvalues: 0, with multiplicity 1, and 1, with multiplicity $k-1$. Therefore, if the laws of the errors also satisfy $a_1 = \ldots = a_k$ (for instance, if all errors have the same distribution), then $4(\theta a_1)^{-1} T^1_{CM} \xrightarrow{\mathcal{L}} \chi^2_{k-1}$, which coincides with the null distribution of the classical Levene's test for equality of variances in two or more groups. To get a consistent null distribution estimator of T^1_{CM} in this case, it suffices to estimate θ and a_1 consistently which is an easy task. The same is true for T_1 if $b_1 = \ldots = b_k$.

For T_{CM}^2 and T_2, the story is somewhat different. From Corollary 5, $4T_{CM}^2$ converges in law to $W_3 = \sum_{j=1}^{k} \gamma_j \chi_{1,j}^2$, where $\chi_{1,1}^2, \ldots, \chi_{1,k}^2$ are as before and $\gamma_1, \ldots, \gamma_k$ are the eigenvalues of $\mathcal{C}\Sigma$, $\mathcal{C} = diag(p)\mathcal{M}diag(p)$, with p as defined in (9) and $\mathcal{M} = (m_{rs})$, $m_{rs} = \sum_{v=1}^{k} p_v \int t^2 f_{\varepsilon_r}(t) f_{\varepsilon_s}(t) f_{\varepsilon_v}(t) dt$, $1 \leq r, s, \leq k$. Note that if Σ is as in (9), since $(I_k - pp')p = 0$ we have that trace$(\mathcal{C}\Sigma) = 0$, and thus $W_3 = 0$. That is to say that, in this case, the asymptotic null distribution of T_{CM}^2 degenerate. The same happens to T_2. Since in practice Σ is unknown, in order to estimate the null distribution of T_{CM}^2 and T_2 it is preferable to use the bootstrap procedure mentioned in the first paragraph of this subsection.

6 Finite sample performance

This section is devoted to the study of the practical performance of the proposed test statistics in terms of level approximation and terms of power. With that purpose, we consider the following variance models in a two-population ($k = 2$) framework:

(L1) $\sigma_1^2(x) = \sigma_2^2(x) = 0.25$

(L2) $\sigma_1^2(x) = \sigma_2^2(x) = \left(\frac{7}{6}0.50x + \frac{1}{2}0.50\right)^2$

(P1) $\sigma_1^2(x) = 0.25$; $\sigma_2^2(x) = 0.50$

(P2) $\sigma_1^2(x) = 0.25$; $\sigma_2^2(x) = 0.75$

(P3) $\sigma_1^2(x) = 0.25$; $\sigma_2^2(x) = \left(\frac{7}{8}\sqrt{0.50}x + \frac{1}{2}\sqrt{0.50}\right)^2$

(P4) $\sigma_1^2(x) = \left(\frac{7}{6}0.50x + \frac{1}{2}0.50\right)^2$; $\sigma_2^2(x) = \left(\frac{7}{8}\sqrt{0.50}x + \frac{1}{2}\sqrt{0.50}\right)^2$

Models (L1) and (L2) are under the null hypothesis, so they will be used to study the level approximation. On the other hand, the power will be investigated through models (P1), (P2), (P3) and (P4). In all cases the regression functions are $m_1(x) = m_2(x) = x$. The distribution of the covariates X_1 and X_2 are $Beta(1.5, 2)$ and $Beta(2, 1.5)$, respectively, and the regression errors ε_1 and ε_2 are $N(0, 1)$. The weight function w required to construct the ECF-based test statistics is the density of a standard normal. The tables will display the observed proportion of rejections in 1000 simulated data sets with significance level $\alpha = 0.05$ (other significance levels were also considered and similar results were obtained).

Nonparametric estimation of the regression functions is performed by local-linear estimation, while the estimation of the conditional variance functions is done with the local-constant (Nadaraya-Watson) estimator, as it guarantees the positiveness of the estimation. The application of these smoothing techniques requires the specification of a smoothing parameter or bandwidth. The choice of this quantity in testing frameworks is not a solved problem (see, for example, the discussion about this topic in González-Manteiga and Crujeiras, 2013). To study the impact of the smoothing parameters in our tests, we will show results obtained under fixed values and also for values obtained by cross-validation. From some unreported simulations, we have learned that taking the same bandwidth in all populations is recommended. In the case of the cross-validation (indicated by cv in the tables), the regular least-squares method was applied to find the smoothing parameters to estimate σ_j^2, $j = 1, 2$, and then the average of the two obtained quantities is used to perform the estimation. A similar procedure is used to obtain the cross-validation bandwidth to estimate the regression functions m_j. On the other hand, in the case of fixed bandwidths we take values 0.1, 0.2 and 0.3 (recall that the support of the covariates is [0,1]) to estimate both the regression and the variance functions.

We first study the behaviour of the tests based on the approximation of the asymptotic null distribution. We will only study the tests based on T_{CM}^1 and T_1 because, as explained in Section 5.1, the asymptotic null distribution of the statistics is a non-degenerate combination of chi-square random variables. Since we are dealing with approximations based on asymptotics, we consider moderate sample sizes (100 and 200). The obtained results are displayed in Table 1. In terms of level approximation, the behaviour of both statistics is reasonable for model (L1). For model (L2), the level is clearly overestimated for samples sizes (100,100), specially in the case of T_{CM}^1. The approximation improves as the sample sizes increase. In terms of power, both statistics present a similar behaviour. The choice of the smoothing parameter does not seems to have an important impact, neither in the approximation of the level, nor in the values of the power.

[Table 1 to be placed around here]

Another possibility to obtain critical values is by means of bootstrap. In particular, in the current setup, as in other related papers (see, for example, Pardo-Fernández *et al.*, 2007, or Dette *et al.*, 2009) a smoothed bootstrap of residuals is recommended. We have applied this bootstrap mechanism with 200 bootstrap replications to the six test statistics proposed in Section 2. Tables 2 and 3 display the observed rejection probabilities for the

ECDF-based tests and for the ECF-based tests, respectively. In this case, smaller sample sizes (50 and 100) are employed. The approximation of the level (models L1 and L2) is good for the tests statistics based on L_2-distances T^1_{CM}, T^2_{CM}, T_1 and T_2. On the other hand, the Kolmogorov-Smirnov type statistics are a bit conservative, specially T^2_{KS}. Regarding the power, the first versions of the test statistics, T^1_{CM}, T^1_{KS} and T_1, achieve better results than T^2_{CM}, T^2_{KS} and T_2, respectively. Moreover, the tests based on L_2-distances (for example, T^1_{CM} and T_1) produce very similar results, and they outperform the Kolmogorov-Smirnov-type statistics. As before, the choice of the smoothing parameters does not have much impact on the rejection frequencies.

[Table 2 to be placed around here]

[Table 3 to be placed around here]

7 Application to data

To illustrate our testing procedure we will use a data set concerning monthly expenditures of several Dutch households. The variable 'log of the total monthly expenditure' is considered as a covariate and 'log of the expenditure on food' is considered as the response. See Einmahl and Van Keilegom (2008) or Pardo-Fernández *et al.* (2007) for more details on these data. In the latter paper the equality of the regression curves of households of 2, 3 and 4 members was tested and the equality between the regression curves of 3-member households (43 observations) and 4-member households (73 observations) was accepted. Here we move one step forward in the comparison of the regression models and test for the equality of the conditional variance functions. Table 4 shows the p-values obtained from the asymptotic null distribution for T_1 and T^1_{CM} or by bootstrap for the six test statistics with fixed bandwidths ranging from 0.20 and 0.50 (the support of the covariates is approximately between 9.5 and 11.5). The results are quite homogeneous, as all test statistics, except the asymptotic version of T^1_{CM}, lead to the acceptance of the equality of the conditional variance functions. As we have seen in the simulations presented in Section 6, the approximation of the asymptotic null distribution of the T^1_{CM} is not satisfactory, specially for small sample sizes. Since here we are working with 43 and 73 observations, the results for this test statistic are not reliable, and we should only consider its bootstrap version.

[Table 4 to be placed around here]

8 Conclusions

In this paper, we constructed and studied six tests for the equality of k conditional variances. To do so, we compared the ECDF and ECF of the error terms estimated nonparametrically under H_0 and H_1. Under some regularity conditions, the proposed tests are consistent against any fixed alternative and are able to detect contiguous alternatives converging to the null at a rate $n^{-1/2}$. The assumptions needed to drive these properties are weaker for the for ECF-based test statistics. Specifically, no requirement is imposed on the distributions of the errors. An approximation of the asymptotic null distribution has been proposed and the performance of each test has been evaluated by means of some simulations. The proposed approximation works, in the sense of providing type I errors close to the nominal values, specially when the sample sizes are at least 100. For smaller sample sizes it is recommended to approximate the null distribution through a bootstrap mechanism.

9 Appendix

We now sketch the proofs of the results stated in Sections 1–4. With this aim we first give some preliminary results, some of them are of independent interest.

9.1 Preliminary results

Under Assumption (A1), and consequently under Assumption A, we have that, for $1 \leq j \leq k$, $\sup_{x\in R} |\hat{m}_j(x) - m_j(x)| = o_p(n_j^{-1/4})$, $\sup_{x\in R} |\hat{\sigma}_j(x) - \sigma_j(x)| = o_p(n_j^{-1/4})$, and $\sup_{x\in R} |\hat{f}_j(x) - f_j(x)| = o_p(n_j^{-1/4})$. This together with some routine calculations show that

$$\sup_{x\in R}\left|\hat{\sigma}_j^2(x) - \sigma_j^2(x) - \frac{1}{n_j f_j(x)}\sum_{s=1}^{n_j} K_h(X_{js}-x)\left[\{Y_{js}-m_j(x)\}^2 - \sigma_j^2(x)\right]\right| = o_p(n^{-1/2}). \tag{10}$$

Also, from the equality,

$$\hat{\sigma}_j(x) - \sigma_j(x) = \frac{\hat{\sigma}_j^2(x) - \sigma_j^2(x)}{2\sigma_j(x)} - \frac{\{\hat{\sigma}_j(x) - \sigma_j(x)\}^2}{2\sigma_j(x)},$$

it follows that,

$$\sup_{x\in R}\left|\hat{\sigma}_j(x) - \sigma_j(x) - \frac{\hat{\sigma}_j^2(x) - \sigma_j^2(x)}{2\sigma_j(x)}\right| = o_p(n^{-1/2}). \tag{11}$$

So, by the definition of $\hat{\sigma}_0(x)$ and $\sigma_0(x)$, we also have that $\sup_{x\in R}|\hat{\sigma}_0(x)-\sigma_0(x)| = o_p(n^{-1/4})$, and

$$\sup_{x\in R}\left|\hat{\sigma}_0(x)-\sigma_0(x)-\sum_{j=1}^{k}\pi_j(x)\frac{\hat{\sigma}_j^2(x)-\sigma_j^2(x)}{2\sigma_0(x)}\right| = o_p(n^{-1/2}). \tag{12}$$

Lemma 9 *Suppose that Assumption (A1) holds. Then,*

(i) $$\int \frac{\hat{\sigma}_j(x)-\sigma_j(x)}{\sigma_j(x)}f_j(x)dx = \frac{1}{2n_j}\sum_{s=1}^{n_j}\left(\varepsilon_{js}^2-1\right)+o_p(n^{-1/2}).$$

(ii) $$\int \frac{\hat{\sigma}_0(x)-\sigma_0(x)}{\sigma_0(x)}f_j(x)dx = \sum_{v=1}^{k}\frac{1}{2n_v}\sum_{s=1}^{n_v}\pi_v(X_{vs})\frac{f_j(X_{vs})}{f_v(X_{vs})}\frac{\sigma_v^2(X_{vs})}{\sigma_0^2(X_{vs})}\left(\varepsilon_{vs}^2-1\right)+o_p(n^{-1/2}).$$

Proof From (10) and (11), we get

$$\int \frac{\hat{\sigma}_j(x)-\sigma_j(x)}{\sigma_0(x)}f_j(x)dx = \frac{1}{2n_j}\sum_{s=1}^{n_j}\int K_h(X_{js}-x)\left[(Y_{js}-m_j(x))^2/\sigma_j^2(x)-1\right]dx+o_p(n^{-1/2}).$$

Part (i) follows from the above equality by making the change of variable $U_{js}=\frac{X_{js}-x}{h}$ and applying Taylor's development. Part (ii) can be proved similarly by using (10) and (12). □

Lemma 10 *Let* $\tilde{\varphi}_{\varepsilon_j}(t)=\frac{1}{n_j}\sum_{l=1}^{n_j}\exp(\mathrm{i}t\varepsilon_{jl})$, $\hat{\varphi}_{\varepsilon_j}(t)=\frac{1}{n_j}\sum_{l=1}^{n_j}\exp(\mathrm{i}t\hat{\varepsilon}_{jl})$, *and similarly, define* $\tilde{\varphi}_{\varepsilon_{0j}}(t)$ *and* $\hat{\varphi}_{\varepsilon_{0j}}(t)$. *Suppose Assumption (A1) holds. Then,*

(i)

$$\hat{\varphi}_{\varepsilon_j}(t)=\tilde{\varphi}_{\varepsilon_j}(t)+\mathrm{i}\frac{t}{n_j}\sum_{l=1}^{n_j}\exp(\mathrm{i}t\varepsilon_{jl})\frac{m_j(X_{jl})-\hat{m}_j(X_{jl})}{\sigma_j(X_{jl})}$$
$$+\mathrm{i}\frac{t}{n_j}\sum_{l=1}^{n_j}\exp(\mathrm{i}t\varepsilon_{jl})\frac{\sigma_j(X_{jl})-\hat{\sigma}_j(X_{jl})}{\sigma_j(X_{jl})}\varepsilon_{jl}+tR_{j,1}(t)+t^2R_{j,2}(t),$$

with $\sup_t|R_{j,s}(t)|=o_p(n^{-1/2})$, $s=1,2$.

(ii)

$$\hat{\varphi}_{\varepsilon_{0j}}(t)=\tilde{\varphi}_{\varepsilon_{0j}}(t)+\mathrm{i}\frac{t}{n_j}\sum_{l=1}^{n_j}\exp(\mathrm{i}t\varepsilon_{0jl})\frac{m_j(X_{jl})-\hat{m}_j(X_{jl})}{\sigma_0(X_{jl})}$$
$$+\mathrm{i}\frac{t}{n_j}\sum_{l=1}^{n_j}\exp(\mathrm{i}t\varepsilon_{0jl})\frac{\sigma_0(X_{jl})-\hat{\sigma}_0(X_{jl})}{\sigma_0(X_{jl})}\varepsilon_{0jl}+tR_{0j,1}(t)+t^2R_{0j,2}(t),$$

with $\sup_t|R_{0j,s}(t)|=o_p(n^{-1/2})$, $s=1,2$.

Proof Using Taylor's development, we get

$$\hat{\varphi}_{\varepsilon_j}(t) - \tilde{\varphi}_{\varepsilon_j}(t) = \mathrm{i}\frac{t}{n_j}\sum_{l=1}^{n_j}(\hat{\varepsilon}_{jl} - \varepsilon_{jl})\exp(\mathrm{i}t\varepsilon_{jl}) + t^2 R_j(t)\frac{1}{n_j}\sum_{l=1}^{n_j}(\hat{\varepsilon}_{jl} - \varepsilon_{jl})^2,$$

with $\sup_t |R_j(t)| = O_p(1)$. Part (i) follows from the following equality,

$$\begin{aligned}\hat{\varepsilon}_j - \varepsilon_j &= \frac{m_j(X_j) - \hat{m}_j(X_j)}{\hat{\sigma}_j(X_j)} + \frac{\sigma_j(X_j) - \hat{\sigma}_j(X_j)}{\hat{\sigma}_j(X_j)}\varepsilon_j \\ &= \frac{m_j(X_j) - \hat{m}_j(X_j)}{\sigma_j(X_j)} + \frac{\{m_j(X_j) - \hat{m}_j(X_j)\}\{\sigma_j(X_j) - \hat{\sigma}_j(X_j)\}}{\sigma_j(X_j)\hat{\sigma}_j(X_j)} \\ &+ \frac{\sigma_j(X_j) - \hat{\sigma}_j(X_j)}{\sigma_j(X_j)}\varepsilon_j + \frac{\{\sigma_j(X_j) - \hat{\sigma}_j(X_j)\}^2}{\sigma(X_j)\hat{\sigma}_j(X_j)}\varepsilon_j.\end{aligned}$$

Similarly, one can prove (ii). □

Lemma 11 *Let g be a bounded function. Suppose Assumption (A1) holds. Then,*

$$\frac{\mathrm{i}t}{\sqrt{n_j}}\sum_{l=1}^{n_j}\varepsilon_{jl}\exp(\mathrm{i}t\varepsilon_{jl})g(X_{jl})\frac{\hat{\sigma}_v(X_{jl}) - \sigma_v(X_{jl})}{\sigma_v(X_{jl})} = \frac{t}{2}\varphi'_{\varepsilon_j}(t)\frac{\sqrt{n}_j}{n_v}\sum_{s=1}^{n_v}(\varepsilon_{vs}^2 - 1)g(X_{vs})\frac{f_j(X_{vs})}{f_v(X_{vs})} + tR_{j,v}(t),$$

with $\sup_t |R_{j,v}(t)| = o_p(1)$, $1 \leq j, v \leq k$.

Proof From (10) and (11),

$$\frac{\mathrm{i}t}{\sqrt{n_j}}\sum_{l=1}^{n_j}\exp(\mathrm{i}t\varepsilon_{jl})\varepsilon_{jl}g(X_{jl})\frac{\hat{\sigma}_v(X_{jl}) - \sigma_v(X_{jl})}{\sigma_v(X_{jl})} = \frac{\mathrm{i}t}{2}\sqrt{n_j}H_{jv} + tR_1(t),$$

where $\sup_t |R_1(t)| = o_p(1)$ and

$$H_{jv}(t) = \frac{1}{n_j n_v}\sum_{l=1}^{n_j}\sum_{s=1}^{n_v}U_v(X_{jl}, \varepsilon_{jl}; X_{vs}, \varepsilon_{vs}; t), \text{ with}$$

$$U_v(X_1, \varepsilon_1; X_2, \varepsilon_2; t) = \varepsilon_1\exp(\mathrm{i}t\varepsilon_1)\frac{g(X_1)}{f_v(X_1)}K_h(X_1 - X_2)\left[\frac{\{m_v(X_2) + \varepsilon_2\sigma_v(X_2) - m_v(X_1)\}^2}{\sigma_v^2(X_1)} - 1\right].$$

• If $j \neq v$, then, for every t, $H_{jv}(t)$ is a two sample U-statistic of degree $(1,1)$ with kernel $U_v(X_{jl}, \varepsilon_{jl}; X_{vs}, \varepsilon_{vs}; t)$. Its Hájek projection, $H'_{jv}(t)$, is given by

$$H'_{jv}(t) = -\mathrm{i}\varphi'_{\varepsilon_j}(t)\frac{1}{n_v}\sum_{s=1}^{n_v}(\varepsilon_{vs}^2 - 1)g(X_{vs})\frac{f_j(X_{vs})}{f_v(X_{vs})} + R'_{jv}(t)$$

where $\sup_t |R'_{jv}(t)| = O_p(h^2)$. Moreover,

$$var\{H_{jv}(t) - H'_{jv}(t)\} = \frac{1}{n_j n_v}E\{U_h^2(X_j, \varepsilon_j; X_v, \varepsilon_v; t)\} = O(n_j^{-1}n_v^{-1}h^{-1}),$$

Therefore,

$$\sqrt{n_j}H_{jv}(t) = -\mathrm{i}\varphi'_{\varepsilon_j}(t)\frac{\sqrt{n_j}}{n_v}\sum_{s=1}^{n_v}(\varepsilon_{vs}^2-1)g(X_{vs})\frac{f_j(X_{vs})}{f_v(X_{vs})}+R_{jv}(t), \quad \text{with} \quad \sup_t |R_{jv}(t)| = o_p(1). \tag{13}$$

• If $j = v$, then

$$H_{jj}(t) = \frac{K(0)}{n_j^2 h}\sum_{l=1}^{n_j}\varepsilon_{jl}\exp(\mathrm{i}t\varepsilon_{jl})(\varepsilon_{jl}^2-1)\frac{g(X_{jl})}{f_j(X_{jl})} + \frac{n_j-1}{2n_j}H_j(t),$$

where, for every t, $H_j(t)$ is a one sample U-statistic of degree 2 with kernel $U_j(X_{jl}, \varepsilon_{jl}; X_{js}, \varepsilon_{js}; t)+U_j(X_{js}, \varepsilon_{js}; X_{jl}, \varepsilon_{jl}; t)$. Arguments very similar to those employed for the case $j \neq v$ can be used to show that

$$\sqrt{n_j}H_j(t) = -2\mathrm{i}\varphi'_{\varepsilon_j}(t)\frac{1}{\sqrt{n_j}}\sum_{s=1}^{n_j}(\varepsilon_{js}^2-1)g(X_{js}) + R_j(t), \quad \text{with} \quad \sup_t |R_j(t)| = o_p(1).$$

Since,

$$\sqrt{n_j}\frac{K(0)}{n_j^2 h}\left|\sum_{l=1}^{n_j}\varepsilon_{jl}\exp(\mathrm{i}t\varepsilon_{jl})(\varepsilon_{jl}^2-1)\frac{g(X_{jl})}{f_j(X_{jl})}\right| \leq \frac{M}{\sqrt{nh^2}}\frac{1}{n_j}\sum_{l=1}^{n_j}|\varepsilon_{jl}|^3,$$

for some positive constant M, we conclude that $H_{jj}(t)$ also satisfies (13) with $j = v$. This proves the result. □

9.2 Proofs of main results

Proof of Theorem 1 (a) The direct implication is trivial. To prove the converse implication, assume that ε_{0j} and ε_j have the same distribution. They will also share the same moments. Now, because of the independence of ε_j and X_j,

$$E(\varepsilon_{0j}^2) = E(\varepsilon_j^2) \Leftrightarrow E\{\sigma_j^2(X_j)/\sigma_0^2(X_j)\} = 1, \text{ and}$$
$$E(\varepsilon_{0j}^4) = E(\varepsilon_j^4) \Leftrightarrow E\{\sigma_j^4(X_j)/\sigma_0^4(X_j)\} = 1.$$

Hence, $E\left(\frac{\sigma_j^2(X_j)}{\sigma_0^2(X_j)}-1\right)^2 = 0$, for $j = 1, \ldots, k$, and so we deduce that H_0 holds.

(b) Let ε (ε_0) be a random variable with CDF F_ε (F_{ε_0}). As for part (a), using the fact that $E(\varepsilon_j^2) = 1$ and $E(\varepsilon_j^4) = E(\varepsilon_1^4) > 0$, for $j = 1, \ldots, k$,

$$E(\varepsilon_0^2) = E(\varepsilon^2) \Leftrightarrow \sum_j p_j E\{\sigma_j^2(X_j)/\sigma_0^2(X_j)\} = 1, \text{ and}$$
$$E(\varepsilon_0^4) = E(\varepsilon^4) \Leftrightarrow \sum_j p_j E\{\sigma_j^4(X_j)/\sigma_0^4(X_j)\} = 1.$$

Hence, $\sum_{j=1}^{k} p_j E\left(\frac{\sigma_j^2(X_j)}{\sigma_0^2(X_j)} - 1\right)^2 = 0$. Since $p_j > 0$, we conclude that H_0 is true. $\square$

Proof of Theorem 2 From the proof of Theorem 1 in Akritas and Van Keilegom (2001),

$$\begin{aligned}\hat{F}_{\varepsilon_{0j}}(t) = \frac{1}{n_j}\sum_{l=1}^{n_j} I(\varepsilon_{0jl} \le t) + t f_{\varepsilon_{0j}}(t)\int \frac{\hat{\sigma}_0(x) - \sigma_0(x)}{\sigma_0(x)} f_j(x)dx \\ + f_{\varepsilon_{0j}}(t)\int \frac{\hat{m}_j(x) - m_j(x)}{\sigma_0(x)} f_j(x)dx + o_p(n^{-1/2}),\end{aligned} \tag{14}$$

and

$$\begin{aligned}\hat{F}_{\varepsilon_j}(t) = \frac{1}{n_j}\sum_{l=1}^{n_j} I(\varepsilon_{jl} \le t) + t f_{\varepsilon_j}(t)\int \frac{\hat{\sigma}_j(x) - \sigma_j(x)}{\sigma_j(x)} f_j(x)dx \\ + f_{\varepsilon_j}(t)\int \frac{\hat{m}_j(x) - m_j(x)}{\sigma_j(x)} f_j(x)dx + o_p(n^{-1/2}),\end{aligned} \tag{15}$$

uniformly in t, where $f_{\varepsilon_{0j}}$ denotes the density corresponding to $F_{\varepsilon_{0j}}$. The desired results follows directly from (14) and (15). $\square$

Proof of Theorem 4 From the proof of Lemma 1 in Akritas and Van Keilegom (2001), we have that

$$\frac{1}{n_j}\sum_{l=1}^{n_j} I(\varepsilon_{jl} \le t) = \frac{1}{n_j}\sum_{l=1}^{n_j} I(\varepsilon_{0jl} \le t) + F_{\varepsilon_j}(t) - F_{\varepsilon_{0j}}(t) + o_p(n^{-1/2}), \tag{16}$$

uniformly in t. Observe that

$$\frac{\sigma_0(x)}{\sigma_j(x)} = 1 - n^{-1/2}\frac{\sigma_0(x)}{\sigma_j(x)}\delta_j(x) = 1 - n^{-1/2}\delta_j(x) + n^{-1}\frac{\sigma_0(x)}{\sigma_j(x)}\delta_j^2(x).$$

Using this and Taylor's development leads to

$$F_{\varepsilon_{0j}}(t) = E\left[F_{\varepsilon_j}\left(t\frac{\sigma_0(X_j)}{\sigma_j(X_j)}\right)\right] = F_{\varepsilon_j}(t) - n^{-1/2} t f_{\varepsilon_j}(t) E(\delta_j(X_j)) + o(n^{-1/2}), \tag{17}$$

uniformly in t,

$$\sup_t |f_{\varepsilon_j}(t) - f_{\varepsilon_{0j}}(t)| = O(n^{-1/2}), \text{ and } \sup_t |t f_{\varepsilon_j}(t) - t f_{\varepsilon_{0j}}(t)| = o(1). \tag{18}$$

From (14)-(18), after some easy calculation, using the fact that $\sigma_j(X_j)/\sigma_0(X_j) = 1 + n^{-1/2}\delta_j(X_j)$, we obtain that,

$$\sqrt{n_j}\left(\hat{F}_{\varepsilon_j}(t) - \hat{F}_{\varepsilon_{0j}}(t)\right) = p_j^{1/2} t f_{\varepsilon_j}(t) E(\delta_j(X_j)) + \frac{t}{2} f_{\varepsilon_j}(t) Z_{n,j} + o_p(1),$$

uniformly in t, where $Z_{n,j}$ is defined in (5). $\square$

Proof of Theorem 6 First observe that

$$\hat{\varphi}_{\varepsilon_j}(t)-\hat{\varphi}_{\varepsilon_{0j}}(t)=\left[\hat{\varphi}_{\varepsilon_j}(t)-\tilde{\varphi}_{\varepsilon_j}(t)\right]-\left[\hat{\varphi}_{\varepsilon_{0j}}(t)-\tilde{\varphi}_{\varepsilon_{0j}}(t)\right]+\left[\tilde{\varphi}_{\varepsilon_j}(t)-\tilde{\varphi}_{\varepsilon_{0j}}(t)\right]. \qquad (19)$$

We have that

$$\int|\hat{\varphi}_{\varepsilon_j}(t)-\tilde{\varphi}_{\varepsilon_j}(t)|^2w(t)\le\frac{2}{n_j}\sum_l(\hat{\varepsilon}_{jl}-\varepsilon_{jl})^2\int t^2w(t)dt=o_p(1),$$

and, similarly, $\int|\hat{\varphi}_{\varepsilon_{0j}}(t)-\tilde{\varphi}_{\varepsilon_{0j}}(t)|^2w(t)=o_p(1)$. On the other hand,

$$\int\left|\tilde{\varphi}_{\varepsilon_j}(t)-\tilde{\varphi}_{\varepsilon_{0j}}(t)\right|^2w(t)dt=\frac{1}{n_j^2}\sum_{r,s=1}^{n_j}\left\{I_w(\varepsilon_{jr}-\varepsilon_{js})+I_w(\varepsilon_{0jr}-\varepsilon_{0js})-2I_w(\varepsilon_{jr}-\varepsilon_{0js})\right\},$$

with I_w as defined in (8), is a V-statistic of degree 2 with a bounded kernel and thus (see Serfling, 1980) it converges to its expected value which is $\int\left|\varphi_{\varepsilon_j}(t)-\varphi_{\varepsilon_{0j}}(t)\right|^2w(t)dt$. We conclude that $\frac{1}{n}T_1\xrightarrow{p}\tau_1$. The limit of $\frac{1}{n}T_2$ can be derived similarly. □

Proof of Theorem 7 Using Taylor's development, we get

$$\tilde{\varphi}_{\varepsilon_{0j}}(t)-\tilde{\varphi}_{\varepsilon_j}(t)=\mathrm{i}\frac{t}{n_j}\sum_{l=1}^{n_j}(\varepsilon_{0jl}-\varepsilon_{jl})\exp(\mathrm{i}t\varepsilon_{jl})+t^2R_{0j}(t)\frac{1}{n_j}\sum_{l=1}^{n_j}(\varepsilon_{0jl}-\varepsilon_{jl})^2,$$

with $\sup_t|R_{0j}(t)|=O_p(1)$. This together with the fact that

$$\varepsilon_{0j}-\varepsilon_j=\left(\frac{\sigma_j(X_j)}{\sigma_0(X_j)}-1\right)\varepsilon_j=n^{-1/2}\delta_j(X_j)\varepsilon_j, \qquad (20)$$

leads to

$$\sqrt{n_j}(\tilde{\varphi}_{\varepsilon_{0j}}(t)-\tilde{\varphi}_{\varepsilon_j}(t))=p_j^{1/2}t\varphi'_{\varepsilon_j}(t)E(\delta_j(X_j))+tR_{0j}^{(1)}(t)+t^2R_{0j}^{(2)}(t), \qquad (21)$$

with $\sup_t|R_{0j}^{(s)}(t)|=o_p(1), s=1,2$.

Now using (12), (11), (20) and Lemmas 11 and 10, we obtain that

$$\begin{aligned}\left[\hat{\varphi}_{\varepsilon_j}(t)-\tilde{\varphi}_{\varepsilon_j}(t)\right]-\left[\hat{\varphi}_{\varepsilon_{0j}}(t)-\tilde{\varphi}_{\varepsilon_{0j}}(t)\right]&=\mathrm{i}\frac{t}{n_j}\sum_{l=1}^{n_j}\exp(\mathrm{i}t\varepsilon_{jl})\frac{\hat{m}_j(X_{jl})-m_j(X_{jl})}{\sigma_j(X_{jl})}\left(\frac{\sigma_j(X_{jl})}{\sigma_0(X_{jl})}-1\right)\\&+\frac{t}{2}\varphi'_{\varepsilon_j}(t)\sum_{v=1}^{k}\frac{1}{n_v}\sum_{s=1}^{n_v}(\varepsilon_{vs}^2-1)\pi_v(X_{vs})\frac{f_j(X_{vs})}{f_v(X_{vs})}\left(\frac{\sigma_j^2(X_{vs})}{\sigma_0^2(X_{vs})}-1\right)\\&-\frac{t}{2}\varphi'_{\varepsilon_j}(t)\frac{1}{\sqrt{n_j}}Z_{n,j}+tR_{0j,1}^{(2)}(t)+t^2R_{0j,2}^{(2)}(t)\\&=-\frac{t}{2}\varphi'_{\varepsilon_j}(t)\frac{1}{\sqrt{n_j}}Z_{n,j}+tR_{0j,1}^{(2)}(t)+t^2R_{0j,2}^{(2)}(t), \qquad (22)\end{aligned}$$

where $Z_{n,j}$ is given by (5) and $\sup_t|R_{0j,s}^{(2)}(t)|=o_p(n^{-1/2})$, $s=1,2$.

Combining, (19), (21) and (22), we conclude that

$$\sqrt{n_j}(\hat{\varphi}_{\varepsilon_j}(t)-\hat{\varphi}_{\varepsilon_{0j}}(t))=p_j^{1/2}t\varphi'_{\varepsilon_j}(t)E(\delta_j(X_j))-\frac{t}{2}\varphi'_{\varepsilon_j}(t)Z_{n,j}+tR_{1j}(t)+t^2R_{2j}(t),$$

with $\sup_t|R_{sj}(t)|=o_p(1)$, $s=1,2$. □

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Table 1: Observed rejection frequencies in 1000 simulated data sets for the tests based on the critical values obtained from the asymptotic null distribution of T^1_{CM} and T_1.

			T^1_{CM}				T_1			
model	(n_1, n_2)	$h:$	cv	0.10	0.20	0.30	cv	0.10	0.20	0.30
(L1)	$(100, 100)$		0.065	0.069	0.061	0.071	0.046	0.046	0.043	0.046
	$(200, 100)$		0.036	0.037	0.039	0.036	0.047	0.048	0.050	0.049
	$(200, 200)$		0.033	0.033	0.038	0.034	0.050	0.052	0.051	0.052
(L2)	$(100, 100)$		0.102	0.093	0.092	0.092	0.074	0.070	0.075	0.076
	$(200, 100)$		0.040	0.041	0.043	0.046	0.061	0.062	0.063	0.065
	$(200, 200)$		0.056	0.056	0.054	0.055	0.067	0.069	0.070	0.072
(P1)	$(100, 100)$		0.909	0.907	0.902	0.908	0.905	0.907	0.901	0.900
	$(200, 100)$		0.927	0.926	0.923	0.924	0.960	0.957	0.955	0.957
	$(200, 200)$		0.992	0.992	0.992	0.991	0.993	0.993	0.993	0.993
(P2)	$(100, 100)$		0.994	0.991	0.987	0.988	0.999	0.997	0.995	0.995
	$(200, 100)$		0.994	0.992	0.991	0.990	0.995	0.995	0.994	0.994
	$(200, 200)$		0.999	0.997	0.997	0.997	1.000	0.998	0.998	0.997
(P3)	$(100, 100)$		0.776	0.775	0.773	0.778	0.768	0.766	0.770	0.771
	$(200, 100)$		0.783	0.781	0.775	0.780	0.888	0.883	0.880	0.882
	$(200, 200)$		0.934	0.937	0.931	0.932	0.964	0.965	0.964	0.964
(P4)	$(100, 100)$		0.676	0.673	0.664	0.669	0.636	0.631	0.635	0.635
	$(200, 100)$		0.650	0.645	0.636	0.640	0.691	0.681	0.684	0.689
	$(200, 200)$		0.852	0.852	0.854	0.854	0.887	0.887	0.884	0.884

Table 2: Observed rejection frequencies in 1000 simulated data sets for the tests based on the test statistics T^1_{CM}, T^2_{CM}, T^1_{KS} and T^2_{KS}. The critical values obtained by bootstrap.

			T^1_{CM}				T^2_{CM}				T^1_{KS}				T^2_{KS}			
model	(n_1, n_2)	$h:$	cv	0.10	0.20	0.30	cv	0.10	0.20	0.30	cv	0.10	0.20	0.30	cv	0.10	0.20	0.30
(L1)	$(50, 50)$		0.045	0.047	0.053	0.048	0.044	0.037	0.050	0.053	0.032	0.034	0.034	0.037	0.025	0.028	0.026	0.028
	$(100, 50)$		0.038	0.036	0.041	0.036	0.049	0.051	0.039	0.046	0.029	0.038	0.035	0.029	0.035	0.034	0.029	0.020
	$(100, 100)$		0.044	0.043	0.039	0.048	0.045	0.046	0.047	0.047	0.034	0.040	0.040	0.038	0.025	0.028	0.022	0.026
(L2)	$(50, 50)$		0.072	0.068	0.070	0.070	0.058	0.049	0.057	0.063	0.050	0.047	0.051	0.049	0.029	0.028	0.032	0.032
	$(100, 50)$		0.040	0.044	0.045	0.045	0.049	0.046	0.052	0.058	0.038	0.040	0.035	0.043	0.033	0.033	0.029	0.037
	$(100, 100)$		0.058	0.049	0.053	0.058	0.061	0.059	0.070	0.066	0.054	0.045	0.042	0.050	0.046	0.033	0.037	0.048
(P1)	$(50, 50)$		0.549	0.530	0.531	0.536	0.383	0.372	0.388	0.368	0.409	0.409	0.401	0.397	0.210	0.183	0.208	0.208
	$(100, 50)$		0.647	0.657	0.634	0.641	0.573	0.569	0.563	0.568	0.586	0.593	0.593	0.598	0.360	0.358	0.362	0.372
	$(100, 100)$		0.885	0.889	0.874	0.884	0.687	0.691	0.690	0.686	0.802	0.812	0.800	0.809	0.457	0.431	0.452	0.469
(P2)	$(50, 50)$		0.905	0.906	0.911	0.902	0.769	0.771	0.783	0.763	0.788	0.806	0.795	0.784	0.497	0.481	0.515	0.506
	$(100, 50)$		0.951	0.954	0.947	0.944	0.900	0.896	0.893	0.895	0.927	0.934	0.920	0.915	0.754	0.756	0.732	0.732
	$(100, 100)$		0.996	0.995	0.995	0.993	0.974	0.976	0.969	0.973	0.992	0.988	0.988	0.992	0.862	0.871	0.869	0.873
(P3)	$(50, 50)$		0.395	0.397	0.397	0.402	0.289	0.291	0.297	0.287	0.296	0.290	0.301	0.289	0.155	0.127	0.145	0.146
	$(100, 50)$		0.503	0.504	0.501	0.496	0.393	0.417	0.413	0.412	0.454	0.444	0.449	0.433	0.216	0.215	0.212	0.225
	$(100, 100)$		0.705	0.713	0.709	0.707	0.521	0.510	0.517	0.512	0.631	0.634	0.630	0.609	0.269	0.254	0.280	0.262
(P4)	$(50, 50)$		0.326	0.310	0.319	0.308	0.216	0.225	0.221	0.221	0.224	0.239	0.224	0.220	0.099	0.104	0.104	0.101
	$(100, 50)$		0.356	0.359	0.355	0.356	0.318	0.312	0.332	0.336	0.313	0.319	0.326	0.314	0.173	0.183	0.183	0.177
	$(100, 100)$		0.555	0.569	0.574	0.567	0.406	0.394	0.394	0.402	0.464	0.486	0.456	0.470	0.226	0.258	0.240	0.243

Table 3: Observed rejection frequencies in 1000 simulated data sets for the tests based on the test statistics T_1 and T_2. The critical values obtained by bootstrap.

			T_1				T_2			
model	(n_1, n_2)	h :	cv	0.10	0.20	0.30	cv	0.10	0.20	0.30
(L1)	$(50, 50)$		0.046	0.046	0.052	0.045	0.040	0.051	0.045	0.047
	$(100, 50)$		0.035	0.032	0.035	0.031	0.041	0.044	0.041	0.041
	$(100, 100)$		0.040	0.038	0.040	0.041	0.048	0.053	0.046	0.052
(L2)	$(50, 50)$		0.070	0.067	0.076	0.069	0.055	0.061	0.056	0.058
	$(100, 50)$		0.043	0.045	0.043	0.041	0.040	0.041	0.034	0.037
	$(100, 100)$		0.058	0.056	0.052	0.060	0.063	0.063	0.057	0.059
(P1)	$(50, 50)$		0.562	0.566	0.553	0.556	0.361	0.358	0.371	0.367
	$(100, 50)$		0.693	0.686	0.679	0.679	0.466	0.480	0.488	0.490
	$(100, 100)$		0.895	0.896	0.887	0.894	0.543	0.541	0.546	0.547
(P2)	$(50, 50)$		0.928	0.926	0.926	0.917	0.726	0.716	0.738	0.738
	$(100, 50)$		0.968	0.971	0.961	0.956	0.850	0.857	0.869	0.859
	$(100, 100)$		0.997	0.997	0.995	0.993	0.937	0.936	0.935	0.938
(P3)	$(50, 50)$		0.427	0.431	0.424	0.440	0.298	0.294	0.300	0.301
	$(100, 50)$		0.581	0.580	0.565	0.566	0.367	0.356	0.376	0.381
	$(100, 100)$		0.743	0.737	0.744	0.742	0.432	0.421	0.412	0.400
(P4)	$(50, 50)$		0.328	0.326	0.331	0.319	0.194	0.192	0.206	0.203
	$(100, 50)$		0.381	0.380	0.383	0.383	0.238	0.240	0.249	0.248
	$(100, 100)$		0.573	0.574	0.581	0.572	0.306	0.308	0.309	0.313

Table 4: p-values for testing for the equality of the conditional variance functions for the data set concerning expenditures of Dutch households.

	aymptotic		bootstrap					
h	T^1_{CM}	T_1	T^1_{CM}	T^2_{CM}	T^1_{KS}	T^2_{KS}	T_1	T_2
0.20	0.011	0.395	0.207	0.101	0.123	0.135	0.428	0.735
0.25	0.028	0.380	0.321	0.296	0.378	0.350	0.387	0.688
0.30	0.028	0.368	0.349	0.269	0.386	0.302	0.387	0.615
0.35	0.045	0.384	0.417	0.280	0.344	0.263	0.377	0.509
0.40	0.049	0.435	0.443	0.274	0.450	0.247	0.435	0.470
0.45	0.078	0.509	0.588	0.247	0.439	0.241	0.536	0.435
0.50	0.083	0.597	0.602	0.239	0.477	0.237	0.647	0.419