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A separation theorem for the weak S-Convex Orders

DENUIT, M., LIU, L. and J. MEYER

A SEPARATION THEOREM FOR THE WEAK S-CONVEX ORDERS

Michel Denuit
Institut de statistique, biostatistique et sciences actuarielles (ISBA)
Université Catholique de Louvain, B-1348 Louvain-la-Neuve, Belgium
michel.denuit@uclouvain.be

Liqun Liu
Private Enterprise Research Center
Texas A&M University, College Station, TX 77843, USA
lliu@tamu.edu

Jack Meyer
Department of Economics
Michigan State University, East Lansing, MI 48824, USA
jmeyer@msu.edu

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Abstract

The present paper extends to higher degrees the well-known separation theorem decomposing a shift in the increasing convex order into a combination of a shift in the usual stochastic order followed by another shift in the convex order. An application in decision making under risk is provided to illustrate the interest of the result.

Key words and phrases: Integrated right and left tails, upper and lower partial moments, stationary excess operator, Khinchine representation, risk increase, risk aversion.

1 Introduction

Stochastic orderings have been used successfully to solve various problems in applied probability and risk management. Once the validity of such a relation is established, it can be exploited to derive a host of inequalities among various quantities. For a general account of the theory and applications of stochastic orders, we refer the reader, e.g., to Shaked and Shanthikumar (2007).

Higher-degree increasing convex and concave orders are now widely used in applied probability and related fields. Rolski (1976) and Fishburn (1976, 1980) based their definition on iterative integrals of the distribution and survival functions, and then identified classes of real functions that define higher-degree increasing convex and concave orders as integral stochastic order relations. These partial order relations naturally appear in the expected utility theory and are very appealing in mathematical economics and actuarial science. See, e.g., Denuit et al. (2005) for a general presentation.

Higher-degree (or the sth degree, $s = 2, 3, 4, \ldots$) convex and concave orders appear as strengthenings of their higher-degree increasing counterparts when the random variables share the same first $s - 1$ moments. They have been widely studied since Ekern (1980) defined their concave version and Denuit et al. (1998) introduced the convex counterpart.

There are situations where random variables have some moments in common, but not enough to be ordered in the higher-degree convex or concave sense. The weak s-convex order relaxes the equality of the respective $(s-1)$th moments, which considerably increases the applicability of the corresponding stochastic order relation, especially for low values of s. See e.g. Kaas and Hesselager (1995) and Denuit (2002). In the present paper, we aim to extend the separation theorem valid for the increasing convex order (which is the weak 2-convex order) to the weak s-convex orders where $s \geq 3$. To this end, we consider random variables sharing the same first $s - 2$ moments.

The remainder of this paper is organized as follows. Section 2 recalls the definition of the stochastic order relations used in the present work. Section 3 is devoted to the extension of the separation theorem from $s = 2$ to $s = 3$. Given its practical importance, we provide a direct derivation of the result for $s = 3$. This allows us to obtain several properties of independent interest. We then link the construction to the Khinchine representation of distributions which are unimodal at 0. Section 4 aims to generalize the construction to higher degrees by means of the stationary-excess operator. The proof is by recurrence, the initial step corresponding to the case $s = 3$ being covered in Section 3. An application in economics is proposed in Section 5 to illustrate the relevance of the results.

All the random variables considered in this paper are assumed to be non-negative, with support contained in $(0, \infty)$. We write $g^{(n)}$ for the nth derivative of the function g, $n = 1, 2, 3, \ldots$ All the expectations appearing in the text are tacitly assumed to exist.

2 Higher-degree convex and weak convex orders

2.1 Integrated right and left tails

Let us now define the orderings used in this paper. To this end, we need to recall several notions. Let F_X be the distribution function for the non-negative random variable X. Starting from $F_X^{[1]} = F_X$, we define $F_X^{[2]}, F_X^{[3]}, \ldots$ recursively from

$$F_X^{[k+1]}(t) = \int_0^t F_X^{[k]}(\xi)\mathrm{d}\xi = \frac{\mathrm{E}[(t-X)_+^k]}{k!}, \ \ k = 1, 2, \ldots$$

where $\xi_+ = \max\{\xi, 0\}$ denotes the positive part of the real ξ and where the last equality follows from integration by parts. By convention, $(t-x)_+^0 = \mathrm{I}[x \leq t]$ where $\mathrm{I}[\cdot]$ is the indicator function.

Similarly, let $\overline{F}_X^{[1]}$ denote the survival function associated to F_X, i.e. $\overline{F}_X^{[1]} = 1 - F_X$ and we define $\overline{F}_X^{[2]}, \overline{F}_X^{[3]}, \ldots$ recursively from

$$\overline{F}_X^{[k+1]}(t) = \int_t^\infty \overline{F}_X^{[k]}(\xi)\mathrm{d}\xi = \frac{\mathrm{E}[(X-t)_+^k]}{k!}, \ \ k = 1, 2, \ldots$$

By convention, $(x-t)_+^0 = \mathrm{I}[x > t]$.

Starting from $(x-t)^{s-1} = ((x-t)_+ - (x-t)_-)^{s-1}$, we get

$$\mathrm{E}[(X-t)_+^{s-1}] = \sum_{j=0}^{s-1} \binom{s-1}{j} \mathrm{E}[X^j](-t)^{s-j-1} + (-1)^s \mathrm{E}[(t-X)_+^{s-1}]. \tag{2.1}$$

The identity (2.1) is useful to convert a result involving integrated right tails into its analog in terms of integrated left tails, and vice versa.

2.2 Convex orders

Let $s \geq 2$ be an integer and let X and Y be two non-negative random variables with finite $(s-1)$th moments. Then, X is said to precede Y in the s-convex order, denoted as $X \preceq_{s-\mathrm{cx}} Y$, when one of the following equivalent conditions hold true:

(i) $\mathrm{E}[g(X)] \leq \mathrm{E}[g(Y)]$ for all the s-convex functions g, i.e. for all the $s-2$ times continuously differentiable functions g such that $g^{(s-2)}$ is convex.

(ii) $\mathrm{E}[g(X)] \leq \mathrm{E}[g(Y)]$ for all the functions g such that $g^{(s)} \geq 0$.

The s-convex orders can be characterized as follows: $X \preceq_{s-\mathrm{cx}} Y$ holds if, and only if, $\mathrm{E}[X^k] = \mathrm{E}[Y^k]$ for $k = 1, 2, \ldots, s-1 \Leftrightarrow \overline{F}_Y^{[k]}(0) - \overline{F}_X^{[k]}(0) = 0$ for $k = 2, \ldots, s$ and

$$\mathrm{E}[(X-t)_+^{s-1}] \leq \mathrm{E}[(Y-t)_+^{s-1}] \text{ for all } t \geq 0, \tag{2.2}$$

or, equivalently,

$$\mathrm{E}[(t-Y)_+^{s-1}] - \mathrm{E}[(t-X)_+^{s-1}] \begin{array}{l} \geq 0 \text{ for all } t \text{ if } s \text{ is even} \\ \\ \leq 0 \text{ for all } t \text{ if } s \text{ is odd.} \end{array} \tag{2.3}$$

The s-convex orderings have been introduced in Denuit et al. (1998) where formal proofs of these equivalences can be found. Notice that (2.3) follows from (2.2) as

$\mathrm{E}[X^k] = \mathrm{E}[Y^k]$ for $k = 1, \ldots, s-1$

$$\Rightarrow \mathrm{E}[(Y-t)_+^{s-1}] - \mathrm{E}[(X-t)_+^{s-1}] = (-1)^s\big(\mathrm{E}[(t-Y)_+^{s-1}] - \mathrm{E}[(t-X)_+^{s-1}]\big)$$

by (2.1). By convention, $\preceq_{1-\mathrm{cx}}$ corresponds to the usual stochastic order, defined by the dominance of the respective distribution functions. For $s = 2$, $\preceq_{2-\mathrm{cx}}$ is the usual convex order.

The s-concave orders defined by Ekern (1980) coincide with the s-convex ones for odd s but reverse the stochastic inequality for even s. Ekern's (1980) definition includes well-known special cases. Famous examples are the mean preserving increase in risk of Rothschild and Stiglitz (1970) corresponding to $s = 2$, also called the concave order in applied probability, or the increase in downside risk defined by Menezes, Geiss and Tressler (1980) corresponding to $s = 3$ in which mean and variance are kept constant while there is a dispersion transfer from high to low wealth levels. More recently, Menezes and Wang (2005) defined an increase in outer risk corresponding to $s = 4$.

2.3 Weak convex orders

In some applications, the equality of the first $s-1$ moments appears to be quite restrictive and it is useful to relax this constraint. This is why the weak s-convex order has been proposed. Given two non-negative random variables X and Y with finite $(s-1)$th moments, X is said to precede Y in the weak s-convex order, denoted as $X \preceq_{s-\mathrm{wcx}} Y$, when one of the following equivalent conditions hold true:

(i) $\mathrm{E}[g(X)] \leq \mathrm{E}[g(Y)]$ for all the $s-2$ times continuously differentiable functions g such that $g^{(s-2)}$ is non-decreasing and convex.

(ii) $\mathrm{E}[g(X)] \leq \mathrm{E}[g(Y)]$ for all the functions g such that $g^{(s-1)} \geq 0$ and $g^{(s)} \geq 0$.

The weak s-convex order can be characterized as follows:

$$X \preceq_{s-\mathrm{wcx}} Y \Leftrightarrow \begin{cases} \mathrm{E}[X^k] = \mathrm{E}[Y^k] \text{ for } k = 1, 2, \ldots, s-2 \\ \mathrm{E}[(X-t)_+^{s-1}] \leq \mathrm{E}[(Y-t)_+^{s-1}] \text{ for all } t \geq 0. \end{cases} \tag{2.4}$$

Clearly, $X \preceq_{s-\mathrm{wcx}} Y \Rightarrow \mathrm{E}[X^{s-1}] \leq \mathrm{E}[Y^{s-1}]$ so that the equality of $(s-1)$th moments is relaxed into an inequality.

The weak s-convex orderings have been studied by Kaas and Hesselager (1995) as a strengthening of the stop-loss dominance of degree $s-1$ in actuarial science; see also Denuit (2002) for further results. Obviously $X \preceq_{s-\mathrm{cx}} Y \Rightarrow X \preceq_{s-\mathrm{wcx}} Y$ and

$$X \preceq_{s-\mathrm{cx}} Y \Leftrightarrow X \preceq_{s-\mathrm{wcx}} Y \text{ and } \mathrm{E}[X^{s-1}] = \mathrm{E}[Y^{s-1}]. \tag{2.5}$$

By convention, $\preceq_{1-\mathrm{cx}} \Leftrightarrow \preceq_{1-\mathrm{wcx}}$ corresponds to the usual stochastic order. For $s = 2$, $\preceq_{2-\mathrm{wcx}}$ is the increasing convex order, also referred to as the stop-loss order in actuarial studies.

The characterization of the s-convex order in terms of iterated integrals of the survival function has been recalled in (2.2). Such results involve higher-degree stop-loss premiums $\mathrm{E}[(X-t)_+^{s-1}]$ and $\mathrm{E}[(Y-t)_+^{s-1}]$, or upper partial moments. Upper partial moments and lower ones are related through (2.1) so that equivalent characterizations in terms of iterated integrals of the distribution function are easily derived. See (2.3). Let us now provide a result similar to (2.3) for the weak s-convex orders. This is precisely stated next, where we obtain the characterization of the weak s-convex orders in terms of iterated integrals of the respective distribution functions, or lower partial moments.

Proposition 2.1. *Given two non-negative random variables with finite $(s-1)$th moments, $X \preceq_{s-wcx} Y$ if, and only if, $E[X^k] = E[Y^k]$ for $k = 1, \ldots, s-2$ and*

$$E[(t-Y)_+^{s-1}] - E[(t-X)_+^{s-1}] \begin{array}{l} \geq E[X^{s-1}] - E[Y^{s-1}] \text{ for all } t \text{ if } s \text{ is even} \\ \\ \leq E[Y^{s-1}] - E[X^{s-1}] \text{ for all } t \text{ if } s \text{ is odd.} \end{array}$$

Proof. Using (2.1), the characterizing condition (2.4) of the weak s-convex order is equivalent to $\mathrm{E}[X^k] = \mathrm{E}[Y^k]$ for $k = 1, 2, \ldots, s-2$ and

$$\begin{aligned} 0 &\leq \mathrm{E}[(Y-t)_+^{s-1}] - \mathrm{E}[(X-t)_+^{s-1}] \\ &= (-1)^s\big(\mathrm{E}[(t-Y)_+^{s-1}] - \mathrm{E}[(t-X)_+^{s-1}]\big) + \mathrm{E}[Y^{s-1}] - \mathrm{E}[X^{s-1}] \text{ for all } t \geq 0, \end{aligned}$$

which ends the proof. □

As $\mathrm{E}[Y^{s-1}] \geq \mathrm{E}[X^{s-1}]$ must hold when $X \preceq_{s-\mathrm{wcx}} Y$, it is important to notice that the sign of the difference $\mathrm{E}[(t-Y)_+^{s-1}] - \mathrm{E}[(t-X)_+^{s-1}]$ is not controlled under the weak s-convex order. This is in contrast with characterization (2.3) for the s-convex order.

3 Separation theorem for the weak 3-convex order

The separation theorem shows that the increasing convex order relation can be interpreted as being simultaneously "smaller" and "less variable". Precisely, it states that any increasing convex order shift can be decomposed into a first shift in the usual stochastic order followed by a second shift in the convex order, the first one increasing the size of the random variable and the second one its variability. For more details and further references, we refer the reader to Shaked and Shanthikumar (2007) or Denuit et al. (2005).

Let us now formally state the separation theorem. Considering two non-negative random variables X and Y with finite means, the stochastic inequality $X \preceq_{2-\mathrm{wcx}} Y$ holds if, and only if, there exists a random variable Z such that

$$X \preceq_{1-\mathrm{cx}} Z \preceq_{2-\mathrm{cx}} Y.$$

This result is very useful for applications and can easily be established. Indeed, it suffices to define $Z = \max\{X, c\}$ where the constant c is chosen in such a way that

$$\mathrm{E}[Y] = \mathrm{E}[Z] = c + \mathrm{E}[(X-c)_+]. \tag{3.1}$$

Before stating the separation theorem for the $\preceq_{3-\mathrm{wcx}}$ order, let us first establish the following technical results, which appear to be useful for the extension to the third degree. The next result extends the definition of c in (3.1).

Lemma 3.1. *If $E[X^2] \leq E[Y^2]$ then the solution to the equation*

$$E[Y^2] = cE[X] + cE[(X-c)_+] + E[(X-c)_+^2] \tag{3.2}$$

exists and is unique.

Proof. The right-hand side is increasing in c. Its derivative is

$$\begin{aligned} & \mathrm{E}[X] + \mathrm{E}[(X-c)_+] + c\frac{\mathrm{d}}{\mathrm{d}c}\int_c^\infty \mathrm{P}[X>t]\mathrm{d}t + 2\frac{\mathrm{d}}{\mathrm{d}c}\int_c^\infty \left(\int_\xi^\infty \mathrm{P}[X>t]\mathrm{d}t\right)\mathrm{d}\xi \\ = \quad & \mathrm{E}[X] - \mathrm{E}[(X-c)_+] - c\mathrm{P}[X>c], \end{aligned}$$

which is non-negative as

$$\mathrm{E}[X] \quad = \quad \int_0^c \mathrm{P}[X>t]\mathrm{d}t + \int_c^\infty \mathrm{P}[X>t]\mathrm{d}t \geq \mathrm{E}[(X-c)_+] + c\mathrm{P}[X>c].$$

Moreover, there always exists such a c as the right-hand side is unbounded, with a minimum obtained for $c=0$ equal to $\mathrm{E}[X^2] \leq \mathrm{E}[Y^2]$. □

The second property considers the behavior of the second lower partial moment truncated at t divided by t^2 and establishes that this ratio increases in t.

Lemma 3.2. *The function $t \mapsto \frac{E[(t-Y)_+^2]}{t^2}$ is non-decreasing.*

Proof. Let us compute its derivative, equal to

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{E}[(t-Y)_+^2]}{t^2} &= 2\frac{\mathrm{d}}{\mathrm{d}t}\frac{\int_0^t\left(\int_0^\xi F_Y(\eta)\mathrm{d}\eta\right)\mathrm{d}\xi}{t^2} \\ &= 2\frac{t^2\int_0^t F_Y(\eta)\mathrm{d}\eta - 2t\int_0^t\left(\int_0^\xi F_Y(\eta)\mathrm{d}\eta\right)\mathrm{d}\xi}{t^4} \end{aligned}$$

which has the same sign as the function φ defined by

$$\varphi(t) = t\int_0^t F_Y(\eta)\mathrm{d}\eta - 2\int_0^t\left(\int_0^\xi F_Y(\eta)\mathrm{d}\eta\right)\mathrm{d}\xi.$$

The derivative of φ is given by

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\varphi(t) &= \int_0^t F_Y(\eta)\mathrm{d}\eta + tF_Y(t) - 2\int_0^t F_Y(\eta)\mathrm{d}\eta \\ &= tF_Y(t) - \int_0^t F_Y(\eta)\mathrm{d}\eta \geq 0 \end{aligned}$$

so that $\varphi(t) \geq \varphi(0) = 0$. □

We are now ready to state the main result of this section.

Proposition 3.3. *Given two non-negative random variables with finite means and variances, $X \preceq_{3-wcx} Y$ if, and only if, there exists a random variable Z such that*

$$X \preceq_{2-cx} Z \preceq_{3-cx} Y.$$

Proof. The "If" part is straightforward to establish, so we focus on the "Only If" part. Let c be the unique solution to (3.2) and define the distribution function F_Z of Z as

$$F_Z(t) = \begin{cases} \frac{1}{c}\mathrm{E}[(c-X)_+] \text{ for } t < c \\ \\ F_X(t) \text{ for } t \geq c. \end{cases}$$

Notice that $F_Z(c-) \leq F_X(c)$ as

$$F_Z(c-) = \frac{\mathrm{E}[(c-X)_+]}{c} = \frac{1}{c}\int_0^c F_X(t)\mathrm{d}t \leq F_X(c).$$

Hence, Z has an atom at 0 and another atom at c. A direct computation shows that

$$\begin{aligned} \mathrm{E}[Z] &= c\left(F_X(c) - \frac{\mathrm{E}[(c-X)_+]}{c}\right) + \int_c^\infty t\mathrm{d}F_X(t) \\ &= cF_X(c) - \int_0^c F_X(t)\mathrm{d}t + \int_c^\infty t\mathrm{d}F_X(t) \\ &= \mathrm{E}[X], \end{aligned}$$

where the last equality follows from integration by parts.

As the distribution functions F_X and F_Z cross only once, the graph of F_Z lying below the graph of F_X after the unique crossing point, we must have $X \preceq_{2-\mathrm{cx}} Z$. It remains to show that $Z \preceq_{3-\mathrm{cx}} Y$ also holds, that is, $\mathrm{E}[Z^2] = \mathrm{E}[Y^2]$ and $\mathrm{E}[(Z-t)_+^2] \leq \mathrm{E}[(Y-t)_+^2]$ for all t. Clearly,

$$\begin{aligned} \mathrm{E}[Z^2] &= 2\int_0^\infty \int_\xi^\infty \overline{F}_Z(t)\mathrm{d}t\mathrm{d}\xi \\ &= 2\left(\int_0^c \left(\int_\xi^c \left(1 - \frac{1}{c}\mathrm{E}[(c-X)_+]\right)\mathrm{d}t + \int_c^\infty \overline{F}_X(t)\mathrm{d}t\right)\mathrm{d}\xi + \int_c^\infty \int_\xi^\infty \overline{F}_X(t)\mathrm{d}t\mathrm{d}\xi\right) \\ &= 2\left(\int_0^c (c-\xi)\left(1 - \frac{1}{c}\mathrm{E}[(c-X)_+]\right)\mathrm{d}\xi + c\mathrm{E}[(X-c)_+] + \frac{\mathrm{E}[(X-c)_+^2]}{2}\right) \\ &= c^2 - c\mathrm{E}[(c-X)_+] + 2c\mathrm{E}[(X-c)_+] + \mathrm{E}[(X-c)_+^2] \\ &= c\big(c + \mathrm{E}[(X-c)_+] - \mathrm{E}[(c-X)_+]\big) + c\mathrm{E}[(X-c)_+] + \mathrm{E}[(X-c)_+^2] \\ &= \mathrm{E}[(X-c)_+^2] + c\mathrm{E}[(X-c)_+] + c\mathrm{E}[X] \\ &= \mathrm{E}[Y^2] \end{aligned}$$

by definition of c. Let us now establish that the inequality $\mathrm{E}[(Z-t)_+^2] \leq \mathrm{E}[(Y-t)_+^2]$ holds for all t. For $t \geq c$, we obviously have

$$\mathrm{E}[(Z-t)_+^2] = \mathrm{E}[(X-t)_+^2] \leq \mathrm{E}[(Y-t)_+^2].$$

Considering $t < c$, (2.1) indicates that we equivalently have to show that $\mathrm{E}[(t-Z)_+^2] \geq \mathrm{E}[(t-Y)_+^2]$. Now,

$$\mathrm{E}[(t-Z)_+^2] = t^2 F_Z(0) = t^2 \frac{\mathrm{E}[(c-X)_+]}{c}.$$

Identity (2.1) with $s = 2$ gives

$$\begin{aligned}
\mathrm{E}[(c-X)_+] &= c - \mathrm{E}[X] + \mathrm{E}[(X-c)_+] \\
&= c - \mathrm{E}[X] + \frac{1}{c}\big(\mathrm{E}[Y^2] - c\mathrm{E}[X] - \mathrm{E}[(X-c)_+^2]\big),
\end{aligned}$$

where the second equality follows from the definition of c. Hence,

$$\begin{aligned}
\mathrm{E}[(t-Z)_+^2] &= t^2 \left(1 - \frac{\mathrm{E}[X]}{c} + \frac{1}{c^2}\big(\mathrm{E}[Y^2] - c\mathrm{E}[X] - \mathrm{E}[(X-c)_+^2]\big)\right) \\
&= \frac{t^2}{c^2}\big(\mathrm{E}[Y^2] - 2c\mathrm{E}[X] + c^2 - \mathrm{E}[(X-c)_+^2]\big) \\
&\geq \frac{t^2}{c^2}\big(\mathrm{E}[Y^2] - 2c\mathrm{E}[Y] + c^2 - \mathrm{E}[(Y-c)_+^2]\big) \\
&= \frac{t^2}{c^2}\big(\mathrm{E}[(Y-c)^2] - \mathrm{E}[(Y-c)_+^2]\big) \\
&= \frac{t^2}{c^2}\mathrm{E}[(c-Y)_+^2].
\end{aligned}$$

By Lemma 3.2, we know that the ratio $\frac{\mathrm{E}[(t-Y)_+^2]}{t^2}$ increases in t, so that

$$\mathrm{E}[(t-Z)_+^2] \geq \frac{t^2}{c^2}\mathrm{E}[(c-Y)_+^2] \geq \mathrm{E}[(t-Y)_+^2],$$

which ends the proof. □

4 Higher-degree extensions

In order to extend the result of Proposition 3.3 to $s \geq 4$, it is useful to derive it in an alternative way, using the Khinchine representation theorem and the stationary excess operator.

For any non-degenerate non-negative random variable X, with a survival function $\overline{F}_X$, and with a finite mean $\mathrm{E}[X]$, let us define the associated random variable $\widetilde{X}$ whose survival function $\overline{F}_{\widetilde{X}}$ is given by

$$\overline{F}_{\widetilde{X}}(x) = \frac{1}{\mathrm{E}[X]}\int_x^\infty \overline{F}_X(t)\,\mathrm{d}t = \frac{\mathrm{E}[(X-x)_+]}{\mathrm{E}[X]}, \quad x \geq 0.$$

The transformation mapping $\overline{F}_X$ to $\overline{F}_{\widetilde{X}}$ is referred to as the stationary-excess operator in the literature. The random variable $\widetilde{X}$ is often called the stationary forward recurrence time, or the equilibrium age or residual lifetime.

Notice that the stationary-excess operator is not one-to-one. For instance, it maps the Bernoulli distribution with parameter p to the unit uniform distribution for each $p \in (0,1)$.

However, for positive random variables (i.e. $X > 0$ almost surely), this operator is one-to-one; see e.g. Lemma 4 of Lin (1998). Whatever the shape of the probability density function for X, $\widetilde{X}$ always has a decreasing probability density function given by $\overline{F}_X/\mathrm{E}[X]$. The inverse of the stationary excess operator is well defined for distributions with decreasing probability density functions.

The relationship between the moments of X and of $\widetilde{X}$ is given by

$$\mathrm{E}[\widetilde{X}^k] = \frac{\mathrm{E}[X^{k+1}]}{(k+1)\mathrm{E}[X]}, \quad k = 1, 2, \ldots, s-2.$$

Therefore, $\mathrm{E}[X^k] = \mathrm{E}[Y^k]$, $k = 1, 2, \ldots, s-1 \Rightarrow \mathrm{E}[\widetilde{X}^k] = \mathrm{E}[\widetilde{Y}^k]$, $k = 1, 2, \ldots, s-2$. More generally,

$$\mathrm{E}[(\widetilde{X} - t)_+^k] = \frac{\mathrm{E}[(X-t)_+^{k+1}]}{(k+1)\mathrm{E}[X]}.$$

For a proof, see e.g. Hesselager et al. (1997, formula (9))

Denuit et al. (1998) established that transforming X and Y to $\widetilde{X}$ and $\widetilde{Y}$ reduces the degree of the s-convex order. Precisely, for any $s \geq 2$, if $\mathrm{E}[X^k] = \mathrm{E}[Y^k]$ holds for $k = 1, 2, \ldots, s-1$ then

$$X \preceq_{s-\mathrm{cx}} Y \Leftrightarrow \widetilde{X} \preceq_{(s-1)-\mathrm{cx}} \widetilde{Y}. \tag{4.1}$$

Similarly, for any $s \geq 3$, if $\mathrm{E}[X^k] = \mathrm{E}[Y^k]$ holds for $k = 1, 2, \ldots, s-2$ then

$$X \preceq_{s-\mathrm{wcx}} Y \Leftrightarrow \widetilde{X} \preceq_{(s-1)-\mathrm{wcx}} \widetilde{Y}. \tag{4.2}$$

Let us now explain how the random variable Z has been constructed, so that Proposition 3.3 can be extended to higher degrees. If $X \preceq_{3-\mathrm{wcx}} Y$ holds then $\widetilde{X} \preceq_{2-\mathrm{wcx}} \widetilde{Y}$. The separation theorem for the usual increasing convex order ensures that there exists a random variable T such that the stochastic inequalities $\widetilde{X} \preceq_{1-\mathrm{cx}} T \preceq_{2-\mathrm{cx}} \widetilde{Y}$ hold. The idea is now to invert the stationary excess operator to revert to the result stated under Proposition 3.3. However, this operator can be inverted only when the random variables have decreasing densities, which is the case for $\widetilde{X}$ and $\widetilde{Y}$, by construction, but not necessarily for T.

Therefore, the originality of the proof for Proposition 3.3 is to construct a random variable T with a decreasing density, so that T is distributed as $\widetilde{Z}$. We know that if the density of $\widetilde{Z}$ crosses once the density of $\widetilde{X}$, dominating it after the unique crossing point, then $\widetilde{X} \preceq_{1-\mathrm{cx}} \widetilde{Z}$ holds. Accounting for the decreasingness of the densities, the natural candidate for the probability density function of $\widetilde{Z}$ is

$$f_{\widetilde{Z}}(t) = \begin{cases} d \text{ if } 0 \leq t \leq c \\ \\ f_{\widetilde{X}}(t) \text{ if } t > c \end{cases}$$

where the positive constants c and d ensure that $f_{\widetilde{Z}}$ integrates to 1 and that $\mathrm{E}[\widetilde{Z}] = \mathrm{E}[\widetilde{Y}]$ holds. The random variable Z appearing in the proof of Proposition 3.3 corresponds to the inverse of $\widetilde{Z}$ by the stationary-excess operator.

Alternatively, this result can also be obtained as a direct consequence of Khinchine's Theorem stating that every non-negative random variable X has a unimodal distribution with a mode 0 if, and only if, there exist independent random variables U and T_X, such that U is uniformly distributed on $[0,1]$ and the product UT_X has distribution function F_X (see, for example, Theorem 1.3 in Dharmadhikari and Joag-Dev (1988)). Interestingly, whereas X has a decreasing density, the random variable T_X appearing in the representation is not constrained anymore. Therefore, we can directly apply the separation theorem to T_X and find a random variable $T_Z = \max\{T_X, c\}$ with c defined from

$$\mathrm{E}[T_Y] = \mathrm{E}[T_Z] = c + \mathrm{E}[(T_X - c)_+]$$

such that $T_X \preceq_{1-\mathrm{cx}} T_Z \preceq_{2-\mathrm{cx}} T_Y$. The desired random variable Z is then obtained as UT_Z. Since $\mathrm{E}[X] = \mathrm{E}[U]\mathrm{E}[T_X] = \frac{1}{2}\mathrm{E}[T_X]$, it is seen that $\mathrm{E}[T_X] = 2\mathrm{E}[X]$. More generally, $\mathrm{E}[T_X^k] = (k+1)\mathrm{E}[X^k]$.

We are now ready to extend Proposition 3.3 to the whole class of the weak s-convex orders.

Proposition 4.1. *Given two non-negative random variables with finite first $s-1$ moments, $X \preceq_{s-wcx} Y$ if, and only if, there exists a random variable Z such that*

$$X \preceq_{(s-1)-cx} Z \preceq_{s-cx} Y.$$

Proof. The "If" part is straightforward to establish, so we focus on the "Only If" part. The proof is by recurrence on s. We know from Proposition 3.3 that the announced result is valid for $s = 3$. Now, let us assume that it holds for some $s \geq 3$ and let us establish it for $s + 1$. We know from (4.2) that when $X \preceq_{(s+1)-\mathrm{wcx}} Y$ we have

$$\widetilde{X} \preceq_{s-\mathrm{wcx}} \widetilde{Y},$$

where $\widetilde{X}$ and $\widetilde{Y}$ are unimodal at 0, so that they are distributed as $UT_{\widetilde{X}}$ and $UT_{\widetilde{Y}}$, respectively, for some random variables $T_{\widetilde{X}}$ and $T_{\widetilde{Y}}$. The recurrence assumption ensures that there exists a random variable $T_{\widetilde{Z}}$ such that

$$T_{\widetilde{X}} \preceq_{(s-1)-\mathrm{cx}} T_{\widetilde{Z}} \preceq_{s-\mathrm{cx}} T_{\widetilde{Y}}.$$

Hence, there exists a random variable $\widetilde{Z} = UT_{\widetilde{Z}}$ unimodal at 0 such that

$$\widetilde{X} \preceq_{(s-1)-\mathrm{cx}} \widetilde{Z} \preceq_{s-\mathrm{cx}} \widetilde{Y}.$$

The random variable $\widetilde{Z}$ appearing in these stochastic inequalities is the stationary excess transform of some random variable Z. Inverting these relations then yields the announced result. □

Remark 4.2. To establish the result in Proposition 4.1, we have proceeded by iteration. Considering the properties of multiple monotone distributions established by Lefèvre and Loisel (2010, 2013), an alternative approach consists in applying the stationary excess operator several times.

5 Application: The tradeoff between two (higher-degree) risk increases (decreases)

Building decision models that involve tradeoffs is a distinguishing feature of economic analysis. In this section, the random variables X and Y represent two random wealths and we show that the choice between X and Y when $X \preceq_{s-\mathrm{wcx}} Y$ amounts to trading off an $(s-1)$th degree risk increase (decrease) for an sth degree risk increase (decrease), and that if some reference individual is indifferent between X and Y, then the preferences over X and Y of all those individuals who are "sth degree more (or less) risk averse" - a notion to be specified below - than the reference individual can be unambiguously predicted. The analysis in this section generalizes Liu and Meyer (2014), where the authors use the increasing convex order (the special case of the weak s-convex order when $s = 2$) to frame the tradeoff of size for risk.

Let us first recall the definition of sth degree risk aversion. In expected utility theory, individual risk attitudes are described by means of behavioral traits including risk aversion, prudence, temperance and their higher-degree extensions. According to Ekern (1980), a differentiable utility function u exhibits sth degree risk aversion if it fulfills the condition $(-1)^{s+1}u^{(s)} \geq 0$. This risk attitude entails a preference for combining relatively good outcomes with bad ones and can be interpreted as a desire to disaggregate the harms of unavoidable risks and losses (Eeckhoudt and Schlesinger, 2006). The common preferences of all the sth degree risk averse decision makers correspond to the concept of an increase in sth-degree risk introduced by Ekern (1980). For s odd, it coincides with the s-convex order whereas for even s the ranking has to be reversed.

According to Proposition 4.1, $X \preceq_{s-\mathrm{wcx}} Y$ if, and only if, there exists a random variable Z such that $X \preceq_{(s-1)-\mathrm{cx}} Z \preceq_{s-\mathrm{cx}} Y$. To see that the choice between random variables X and Y amounts to trading off an $(s-1)$th degree risk increase (decrease) for an sth degree risk increase (decrease), consider two situations:

(i) Suppose that s is odd. Then $X \preceq_{(s-1)-\mathrm{cx}} Z$ is the same as saying X has less $(s-1)$th degree risk than Z, i.e. X is preferred over Z by every decision maker with utility function u exhibiting $(s-1)$th degree risk aversion. Similarly, and $Z \preceq_{s-\mathrm{cx}} Y$ is the same as saying Y has less sth degree risk than Z, i.e. Y is preferred over Z by every decision maker with utility function u exhibiting sth degree risk aversion. Therefore, choosing between X and Y is about choosing between two risk decreases, X representing the $(s-1)$th degree risk decrease and Y the sth degree risk decrease.

(ii) Suppose that s is even. Then $X \preceq_{(s-1)-\mathrm{cx}} Z$ is the same as saying X has more $(s-1)$th degree risk than Z, and $Z \preceq_{s-\mathrm{cx}} Y$ is the same as saying Y has more sth degree risk than Z. Therefore, choosing between X and Y is about choosing between two risk increases, X representing the $(s-1)$th degree risk increase and Y the sth degree risk increase.

Now assume that all the individuals considered are both $(s-1)$th and sth degree risk averse, i.e. every utility function u satisfies $(-1)^s u^{(s-1)} \geq 0$ and $(-1)^{s+1}u^{(s)} \geq 0$. Intuitively, in situation (i) where the choice is between two risk decreases with Y representing the sth degree risk decrease, if a reference individual is indifferent between X and Y then all those who are "sth degree more risk averse" than the reference individual would prefer Y to X and all those who are "sth degree less risk averse" than the reference individual would prefer X to Y. Similarly, in situation (ii) where the choice is between two risk increases with Y representing the sth degree risk increase, if a reference individual is indifferent between X and Y then all those who are "sth degree more risk averse" than the reference individual would prefer X to Y and all those who are "sth degree less risk averse" than the reference individual would prefer Y to X.

Liu and Meyer (2013) define multiple notions of nth degree more risk aversion, generalizing a line of research that extends Ross' (1981) concept of more risk aversion to higher degrees (see Modica and Scarsini, 2005, Jindapon and Neilson, 2007, Li, 2009 and Denuit and Eeckhoudt, 2010, among others). Let m and n be any two positive integers such that $n > m \geq 1$, and the two utility functions u and v each be both nth degree and mth degree risk averse. Liu and Meyer (2013) give the following definition of the (n/m)th degree Ross more risk aversion: the utility function u is (n/m)th degree Ross more risk averse than the utility function v if

$$\frac{(-1)^{n+1}u^{(n)}(x)}{(-1)^{m+1}u^{(m)}(y)} \geq \frac{(-1)^{n+1}v^{(n)}(x)}{(-1)^{m+1}v^{(m)}(y)} \tag{5.1}$$

for all x and y, or equivalently, if there exists $\lambda > 0$, such that

$$\frac{u^{(n)}(x)}{v^{(n)}(x)} \geq \lambda \geq \frac{u^{(m)}(y)}{v^{(m)}(y)} \tag{5.2}$$

for all x and y. Liu and Meyer (2013) established that (5.1)-(5.2) hold true if, and only if, there exists $\lambda > 0$ and ϕ such that

$$u = \lambda v + \phi, \tag{5.3}$$

where $(-1)^{m+1}\phi^{(m)} \leq 0$ and $(-1)^{n+1}\phi^{(n)} \geq 0$.

Note that the only possible choice for m is $m = 1$ when $n = 2$, and (2/1)nd degree Ross more risk aversion is exactly the original concept of Ross more risk aversion. When $n > 2$, on the other hand, there are multiple versions of "nth degree more risk aversion", for each $m \in \{1, \ldots, n-1\}$. It turns out that the relevant notion of (higher degree) more risk aversion that corresponds to the weak s-convex order is the $(s/(s-1))$th degree Ross more risk aversion.

We are now in a position to establish the following result that indicates the usefulness of the stochastic relation $X \preceq_{s-\text{wcx}} Y$ in framing the tradeoff between two (higher degree) risk increases (decreases).

Proposition 5.1. *Suppose that $X \preceq_{s-wcx} Y$ and that for some utility function v, $E[v(X)] = E[v(Y)]$.*

(i) If s is odd, then for any u that is $(s/(s-1))$th degree Ross more (less) risk averse than v, $E[u(Y)] \geq (\leq) E[u(X)]$.

(ii) If s is even, then for any u that is $(s/(s-1))$th degree Ross more (less) risk averse than v, $E[u(Y)] \leq (\geq) E[u(X)]$.

Proof. There are totally four cases depending on whether s is odd or even and whether u is $(s/(s-1))$th degree Ross more or less risk averse than v. We need only prove the announced result for the case where s is odd and u is $(s/(s-1))$th degree Ross more risk averse than v because the proofs for other cases are very similar. Suppose that s is odd and that u is $(s/(s-1))$th degree Ross more risk averse than v. Then, according to (5.3), there exists $\lambda > 0$ and ϕ such that $u = \lambda v + \phi$, where $\phi^{(s-1)} \geq 0$ and $\phi^{(s)} \geq 0$. Therefore,

$$\begin{aligned}
\mathrm{E}[u(Y)] &= \lambda \mathrm{E}[v(Y)] + \mathrm{E}[\phi(Y)] \\
&= \lambda \mathrm{E}[v(X)] + \mathrm{E}[\phi(Y)] \\
&\geq \lambda \mathrm{E}[v(X)] + \mathrm{E}[\phi(X)] \\
&= \mathrm{E}[u(X)]
\end{aligned}$$

where the inequality is due to $X \preceq_{s-\mathrm{wcx}} Y$, and $\phi^{(s-1)} \geq 0$ and $\phi^{(s)} \geq 0$. This ends the proof. □

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