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Automatic balancing mechanisms for
Notional Defined Contribution Accounts in the
presence of uncertainty

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AUTOMATIC BALANCING MECHANISMS FOR NOTIONAL DEFINED CONTRIBUTION ACCOUNTS IN THE PRESENCE OF UNCERTAINTY

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Abstract

The notional defined contribution model combines pay-as-you-go financing and a defined contribution pension formula. This paper aims to demonstrate the extent to which liquidity and solvency indicators are affected by fluctuations in economic and demographic conditions and to explore the introduction of an automatic balancing mechanism into the pension system. We demonstrate that the introduction of an automatic balancing mechanism reduces the volatility of the buffer fund and that, in most cases, the automatic mechanism that re-establishes solvency produces the lowest variance of the notional factor with the lowest expected value.

JEL: E62, H55, J26

Keywords: actuarial analysis, public pensions, retirement, solvency, stochastic processes, risk.

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Introduction

The introduction of Notional (non-financial) Defined Contribution (NDC) pension accounts as components of modern, multi-pillar pension systems in some countries has been a major innovation of the last two decades of pension reform. NDCs have been established in Italy (1995), Latvia (1996), Poland (1999), Sweden (1999) and Brazil (1999)¹. They are also found in Kyrgyzstan (1997), Mongolia (2000) and Russia (2002), while Egypt combines a notional defined contribution and financial defined contribution system for new entrants after 2013. According to Holzmann et al. (2012), countries such as China and Greece are also seriously considering introducing notional defined contribution pension schemes. Holzmann (2006) argues that notional accounts should provide the foundation for a unified pension system in the European Union.

A notional model is based on a pay-as-you-go (PAYG) system that deliberately mimics a defined contribution pension scheme in which the pension depends on both contributions and investment returns, that is, the capital accumulated over the course of participant employment. The returns on contributions are calculated utilizing a notional rate that reflects the financial health of the system, which is linked to an external index set by law, such as the growth rate of GDP, average wages, or contribution payments. The account balance is called notional because it is utilized only for record keeping, i.e., the system does not invest funds in financial markets as the system is based on pay-as-you-go financing. Nevertheless, when an individual reaches retirement age, the accumulated capital is converted into an annuity that considers the life expectancy of the individual, indexation and technical interest rate. Theoretically, the contribution rate is fixed in a notional defined contribution pension system.

Notional defined contribution systems achieve stability mainly because of two reasons. First, the return on contributions reflects their pay-as-you-go nature (rather than market-based returns), and second, the pension annuity, which is calculated at retirement, should incorporate life expectancy improvements². However, Valdés-Prieto (2000) demonstrates that notional benefit formulae cannot provide automatic equilibrium over the short run and suggests conditions for the pension system to achieve long-run stability.

Auerbach and Lee (2007) observe that the basic notional defined contribution scheme is effective at preventing excessive debt accumulation but not at preventing considerable asset accumulation over time. The authors also demonstrate that the notional defined contribution approach can be more stable with certain adjustments.

According to Boado-Penas et al. (2008), Sweden has gone beyond the notional defined contribution pension system and compiles an actuarial balance sheet since 2006. This balance sheet not only addresses a social demand for transparency in the management of public finances but also decreases the political risk of the pay-as-you-go aspects and increases credibility that promised payments will be rationally respected. Sweden has also implemented an automatic balance mechanism (Vidal-

¹See Vidal-Meliá et al. (2006) for the main features of their notional defined contribution pension system.

²NDC pension systems do not fully address the problem of aging. Although NDCs take the evolution of mortality into account, there is a delay before it happens.

Meliá et al., 2009) to adapt the pension system to changes in socioeconomic and demographic conditions. Some countries with classical Defined Benefit (DB) pension systems, such as Germany, Austria, France, Finland, and Portugal, have also incorporated adjustment mechanisms to assist in calculating or indexing the initial pension payment to mitigate demographic changes.

When designing a pension system, a legislator can obtain two types of equilibria (Bommier and Lee, 2003). The first equilibrium is cross-sectional, while the other is longitudinal. A cross-sectional equilibrium implies that all that is received is given. For example, in a pure pay-as-you-go pension system, pensions for retirees are paid for by the contributions of the working-age population. A longitudinal equilibrium corresponds to a cohort's lifecycle, and an individual or a cohort cannot receive more than his or her own contribution. This equilibrium can also be called actuarial fairness, which means that, at any moment, the present value of lifetime contributions equals the present value of lifetime benefits. A notional defined contribution pension system depends on compounded lifetime contributions, cohort life expectancy at the time of retirement, indexation and technical interest rate. This system can thus be considered actuarially fair at some extent (see Palmer (2006) and Queisser and Whitehouse (2006)). Nevertheless, the underlying design does not produce an automatic cross-sectional equilibrium.

This paper aims to demonstrate how liquidity and solvency indicators in an actuarially fair notional defined contribution system are affected by fluctuations in economic and demographic conditions and to explore the introduction of an automatic balancing mechanism into a notional model to re-establish financial equilibrium.

This paper is structured as follows. Following this brief introduction, the subsequent section, 'A dynamic four overlapping generations model', presents NDCs in a framework with four overlapping generations that allows a dynamic evolution of the main variables that affect the liquidity and solvency indicators of the system. The next section, 'Liquidity and solvency indicators', provides two different methods of assessing the financial health of a pension system. The section, 'Design of an Automatic Balance Mechanism (ABM)', focuses on the design of an automatic balancing mechanism to restore the liquidity or solvency of the system. The final section of this paper provides the main conclusions, while four appendices provide additional details.

A dynamic four overlapping generations model

This section examines NDCs utilizing a four overlapping generations (OLG) model that includes the dynamic population and salary values. Within a four Overlapping Generations (OLG) model, two generations³ of contributors aged y and $y + 1$ and

³The terms 'generation' and 'cohort' are often used synonymously, and, while in some cases this may be appropriate, their meaning should be clarified. A cohort is defined as a group of people sharing a common temporal demographic or economic experience who are observed over time. For example, the retirement cohort of 2012 includes the individuals who retired in that year. A generation refers to all the individuals born in the same year, i.e., a birth cohort, and one can identify different cohorts within the same generation. This paper considers four generations (or birth cohorts) and a retirement age fixed at $y + 3$.

Figure 1: A four overlapping generations framework at time t

Note: The figure illustrates the four-period overlapping generations dynamics at time t when two generations of contributors and two generations of pensioners coexist.

two generations of pensioners aged $y + 2$ and $y + 3$ coexist (see Figure 1 for details). The longest possible lifespan is $y + 4$. The choice of four generations is not arbitrary; it introduces heterogeneity among contributors because these two generations are characterized by different demographic and salary histories as well as contributions to the system. A four-generation model introduces both mortality and indexation during the retirement period.

Population and salary dynamics

The demographic-economic structure at any time t is represented as follows:

Age:

$y, y + 1, y + 2, y + 3$

Population at time t :

$$l(x, t) = l(y, t - x + y)p(x, t) = l(y, 0) \exp \left(\sum_{i=1}^{t-x+y} R_i \right) p(x, t) \quad (1)$$

where

$l(x, t)$ denotes individuals aged x for $x = y, y + 1, y + 2$ and $y + 3$ who are alive at time $t > 0$ and entered the labor market at time $t - x + y$. New entrants at time t are denoted as $l(y, t)$. In this model, the population is supposed given at time $t = 0$.

$p(x, t)$ is the time-dependent survival probability, that is, that an individual will reach age x by time t . It is assumed that no mortality occurs prior to retirement; therefore, $p(x, t) = 1$ for $x = y, y + 1$, and $y + 2$. Furthermore, the survival probability $p(y + 4, t)$ is zero. For simplicity, the survival probability between the first and second retirement period is denoted by $p(y + 3, t) = p_t$.

R_i is the rate of population variation from period $i-1$ to period i , which is a stochastic process defined in the probability space $(\Omega, \mathcal{F}, P)$.

Table 1 illustrates how the population evolves over the first four periods by year of birth.

Salaries at time t

Table 1: Population evolution in terms of entry age y

Age \ Time	$t + 3$	$t + 4$	$t + 5$	$t + 6$
y	$l(y, t + 2)e^{R_{t+3}}$	$l(y, t + 3)e^{R_{t+4}}$	$l(y, t + 4)e^{R_{t+5}}$	$l(y, t + 5)e^{R_{t+6}}$
$y + 1$	$l(y, t + 1)e^{R_{t+2}}$	$l(y, t + 2)e^{R_{t+3}}$	$l(y, t + 3)e^{R_{t+4}}$	$l(y, t + 4)e^{R_{t+5}}$
$y + 2$	$l(y, t)e^{R_{t+1}}$	$l(y, t + 1)e^{R_{t+2}}$	$l(y, t + 2)e^{R_{t+3}}$	$l(y, t + 3)e^{R_{t+4}}$
$y + 3$	$p_{t+3}l(y, t)$	$p_{t+4}l(y, t)e^{R_{t+1}}$	$p_{t+5}l(y, t + 1)e^{R_{t+2}}$	$p_{t+6}l(y, t + 2)e^{R_{t+3}}$
$y + 4$	0	0	0	0

Source: the authors.

$$S(x, t) = S(x, 0) \exp \sum_{i=1}^t \gamma_i \quad (2)$$

where

$S(x, t)$ denotes the salaries⁴ for $x = y, y + 1$, which are earned by the active population. Salaries are dependent on time t and age x . Furthermore, $S(x, 0)$ is the observed salary at time 0 for each active age $x = y, y + 1$.

γ_i is the rate of salary variation from the period $i - 1$ to period i , which is a stochastic process defined in the probability space $(\Omega, \mathcal{F}, P)$.

Liquidity and solvency indicators

There are various methods of assessing the financial health of an unfunded pension system. This paper focuses on two methods. The first, most natural method assesses the liquidity ratio, which indicates the relationship between income from contributions and pension expenditures at a specific date.

A second method assesses the solvency of the pension system by compiling an actuarial balance sheet⁵. This methodology projects changes in the number of contributors and pensioners. The macroeconomic scenario that determines the amount of future contributions and pensions is exogenous. In general, the solvency ratio that emerges from this type of balance is calculated as the difference between the present value of the income from contributions for the next 75-95 years and pension expenditures.

⁴The contribution bases are assumed to coincide with salaries.

⁵According to the Board of Trustees (2013), a detailed actuarial balance has been compiled in the United States since 1941, and since 2002, the balance sheet has included a stochastic methodology. This actuarial balance sheet is similar to that published by Japanese authorities every five years (Sakamoto, 2005) and Canadian authorities every three years (OSFIC 2005, 2007)

In Sweden, compiling an official actuarial balance sheet has been normal practice since 2001. This actuarial balance sheet is a traditional accounting balance sheet that lists assets and liabilities over an indefinite horizon (see Boado-Penas et al. (2008) and Swedish Pension Agency (2013)). The actuarial balance sheet is a financial statement listing the pension system obligations to contributors and pensioners at a particular time and the various assets (financial and contributions) that back these obligations.

The solvency ratio, defined as the ratio of assets to liabilities, calculated from the actuarial balance sheet serves the following two purposes: to measure whether the system can fulfill its obligations to participants and to decide whether an automatic balance mechanism should be applied. The following subsections provide the calculations of the liquidity and solvency ratios.

Liquidity ratio

As previously noted, NDCs are financed on a PAYG basis, that is, retiree pensions are paid for by the contributions of the working age population. The liquidity ratio indicates whether the current contributions and financial assets are sufficient to pay current retiree pensions. Formally, the ratio is represented as follows:

$$LR_t = \frac{C(t) + F^-(t)}{O(t)} \quad (3)$$

with

$$F^-(t) = F(t-1)^+ e^{r_t} = F(0)e^{\sum_{i=1}^t r_i} + \sum_{j=0}^{t-1} (C(j) - O(j))e^{\sum_{i=j+1}^t r_i} \quad (4)$$

where,

r_t represents the financial rate of return of the fund from period $t-1$ to t and is a stochastic process defined in the probability space $(\Omega, \mathcal{F}, P)$.

$F^-(t)$ represents the value of the fund, also called the buffer fund, before new contributions and expenses are considered. The value of the fund after contributions and payments is denoted by $F^+(t) = F^-(t) + C(t) - O(t)$.

$C(t)$ represents income from contributions and $O(t)$ represents pension expenditures for period t .

Income from contributions

The income from contributions received by the pension system at time t , $C(t)$, is represented as follows:

$$\begin{aligned}
C(t) &= \pi \cdot l(y, t) \cdot S(y, t) + \pi \cdot l(y + 1, t) \cdot S(y + 1, t) \\
&= \pi l(y, 0) \exp \left(\sum_{i=1}^t \gamma_i + \sum_{i=1}^{t-1} R_i \right) K_C(t)
\end{aligned} \tag{5}$$

where,

$l(y, t)$ and $l(y + 1, t)$ denote the number of contributors aged y and $y + 1$, respectively.

π is the contribution rate of the pension system.

For simplicity, $S(y, 0)e^{R_t} + S(y + 1, 0)$ is denoted by $K_C(t)$.

The notional compounding factor at time t , $I(t)$, is chosen such that it reflects the financial health of the system through changes in the total contribution basis and is represented as follows:

$$I(t) = \frac{C(t)}{C(t-1)} = e^{\gamma_t + R_{t-1}} \frac{K_C(t)}{K_C(t-1)} = \exp(nr_t) \tag{6}$$

where nr_t is the notional rate for the period $t-1$ to t . Note that this notional factor is affected by both salary and population processes, such as new active individuals or salary trends. This factor corresponds to a notional rate $nr_t = \gamma_t + R_{t-1} + \log \left(\frac{K_C(t)}{K_C(t-1)} \right) - 1$ at time t . In this paper, we will use the notional factor for clarity.

Note that if the population is in a steady state, the notional factor reduces to $I(t) = e^{\gamma + R}$, which means that the notional rate equals $\gamma + R$ and corresponds to the biological interest rate of the economy as indicated in Samuelson (1958). The population is in steady state if it does not depend on time. In our framework, steady state holds if $R_t = R \forall t$ and if the survival probabilities are constant.

Pension expenditures

At retirement, the accumulated notional capital for each individual is converted into an annuity for the next two periods. In this model, all individuals in the same birth cohort have the same contribution history and mortality. Therefore, the accumulated notional capital, $NDC(x, t)$ for $x = y, y + 1, y + 2, y + 3$, will be calculated for each generation⁶.

To calculate the accumulated notional capital we consider past contributions and assign them an interest rate that corresponds to the notional factor (6) until retirement. The total notional accumulated capital for the cohort aged $y + 2$ retiring at time t is represented as follows:

⁶Note that this calculation method allows surviving contributors to receive eventual inheritance gains from the deceased (Boado-Penas and Vidal-Melia, 2014).

$$\text{NDC}(y + 2, t) = C(t)K_N(t) \quad (7)$$

where

$K_N(t) = \frac{S(y,0)e^{R_{t-2}}}{K_C(t-2)} + \frac{S(y+1,0)}{K_C(t-1)}$ is a constant that depends on the notional factor during contributors' career. The notional capital associated with this generation is thus proportional to their current total contributions $C(t)$ to the system. The time-dependent constant $K_N(t)$ equals 1 when $R_{t-2} = R_{t-1}$.

The individual pension $P(y + 2, t)$ to be paid at time t to the new retirees aged $y + 2$ is represented as follows:

$$P(y + 2, t) = \frac{\text{NDC}(y + 2, t)}{a_t l(y + 2, t)} \quad (8)$$

with

$$a_t = \mathbb{E}_t \left[1 + m_{t+1} \frac{\lambda(t + 1)}{i(t + 1)} \right] \quad (9)$$

where

$\text{NDC}(y + 2, t)$ represents the notional accumulated capital of the cohort aged $y + 2$ at time t .

$l(y + 2, t)$ represents the population of new retirees entitled to a pension at time t .

a_t represents the annuity that transforms the accumulated notional capital into a pension considering indexation λ , mortality, m , and discount (or notional factor), i , processes.

m_{t+1} represents the legislator's choice for the mortality rate at time $t + 1$.

$\lambda(t + 1)$ represents the indexation process of pensions at time $t + 1$.

$i(t + 1)$ represents the notional factor process for period $t + 1$.

Calculating the annuity at time t requires assumptions about future processes, which are made in accordance with known information, given by the filtration $\mathcal{F}_t$ at time t .

The pension for the second generation of retirees $P(y + 3, t)$, i.e., those aged $y + 3$, corresponds to the indexed first period pension as follows:

$$P(y + 3, t) = P(y + 2, t - 1)\Lambda^*(t) \quad (10)$$

where $\Lambda^*(t) = \Lambda(t)A_t$ is the adjusted indexation process at time t . The observed value at time t $\Lambda(t)$ of the legislator's indexation process $\lambda(t)$ may differ from the expected value $\mathbb{E}_{t-1}[\lambda(t)]$ considered when the annuity is calculated at time $t - 1$, disturbing the effective longitudinal cohort equilibrium. The adjustment, A_t , ensures actuarial

fairness or longitudinal equilibrium⁷. The adjustment, A_t , is derived in Appendix A and is represented by the following expression:

$$A_t = \frac{I_z(t)}{\Lambda(t)} \frac{1}{p_t} \mathbb{E}_{t-1} \left[m_t \frac{\lambda(t)}{i(t)} \right] \quad (11)$$

where

$I_z(t)$ is the observed value of the notional factor process. This factor can eventually be affected by some pension balancing mechanism z .

p_t and $\Lambda(t)$ denote the observed mortality rate and indexation rate at time t , respectively.

Pension expenditure, which is denoted as $O(t)$, is described by the following expression:

$$O(t) = C(t)K_O(t) \quad (12)$$

where

$K_O(t) = \frac{K_N(t)}{a_t} + K_N(t-1) \frac{a_{t-1}-1}{a_{t-1}}$. The outcome, $O(t)$, is also proportional to the total contribution $C(t)$ paid during the same period.

The notional factor used in this pension design considers demographic and salary changes. Nevertheless, this is insufficient to ensure liquidity (i.e., a balance between contributions and expenses) in this four-period overlapping generations model. This claim has been demonstrated in Valdés-Prieto (2000) for a continuous time model, and the same result is provided in this section.

Proposition 1. *In general, contributions are not equal to pension expenditures in this 4-period overlapping generations unfunded dynamic model, i.e., $C(t) \neq O(t) \forall t$.*

Proof. This result is proven by counterexample. Pension expenditures equal the contributions if and only if $K_O(t) = 1 \forall t$. Assume that $R_t = R = \text{cte}$ and $\gamma_t = \gamma = \text{cte} \forall t$ and deterministic. Thus,

$$K_O(t) = \frac{K_N(t)}{a_t} + K_N(t-1) \frac{a_{t-1}-1}{a_{t-1}} = 1 + \frac{1}{a_t} - \frac{1}{a_{t-1}} \quad (13)$$

$$C(t) = O(t) \Leftrightarrow a_t = a_{t-1} \forall t$$

The contributions equal pension expenditures when the value of annuities is constant at all moments, which only occurs when the system is in the steady state. $\square$

⁷Note that this adjustment would be unnecessary if notional capital at retirement was paid as a lump sum. Furthermore, the adjustment, A_t , allows us to develop a general framework because the expressions in our model remain unchanged regardless of the legislator's process choices about the annuity assumptions.

Accounting solvency ratio

Another method of assessing the health of a pension system is analyzing current liabilities to participants in the pension system compared to current assets. These liabilities can be easily calculated using classical actuarial techniques. In second and third pillar pension systems, these liabilities are normally backed by financial assets that can be valued at any time⁸. However, first pillar pension systems are not generally backed by financial assets. Normally, pension expenditures are financed by current contributions to the pension system, which creates intergenerational solidarity. Current active workers pay current retiree pensions with the promise that future active workers will finance their own pensions.

In the Swedish context, assets are calculated using an accounting measure known as the Contribution Asset. This method of valuating assets was derived for a steady state population. In Sweden, both assets and liabilities are valued based on verifiable cross-section facts, i.e., no projections are made. This should not be interpreted as a belief that all the basic parameters determining the items on the balance sheet will remain constant but as a conscious policy to prefer cross-sectional data. Changes are not included until they occur and can be verified. Then, these changes are incorporated into the balance sheet on an annual basis. Swedish authorities note that system solvency does not depend on either assets or liabilities but on the relationship between these two via the solvency ratio. Therefore, valuing assets and liabilities with cross-sectional data is adequate if applied consistently⁹.

The solvency ratio (in Sweden, balance ratio) is defined as the relationship between assets and liabilities as follows:

$$SR(t) = \frac{CA(t) + F^-(t)}{V(t)} \quad (14)$$

where $F^-(t)$ is given by (4), $CA(t)$ corresponds to the contribution asset and $V(t)$ to the liabilities to all participants in the pension system.

The Contribution Asset

The contribution asset is calculated as the product of the current contribution base $C(t)$ and current turnover duration $TD(t)$ as follows:

$$CA(t) = C(t)TD(t) = C(t) \left(2 + \frac{K_N(t-1)}{K_O(t)} \frac{a_{t-1} - 1}{a_{t-1}} - \frac{S(y+1, 0)}{K_C(t)} \right) \quad (15)$$

⁸Some countries, such as Chile, apply a funding framework to their first pillar pension systems (Fajnzylber and Robalino, 2012).

⁹See Settergren & Mikula (2005) for the notional model, Boado-Penas et al. (2008) and Boado-Penas et al. (2013) for the defined benefit pay-as-you-go system and Bommier and Lee (2003) for a general transfer model under the golden rule.

where

$TD(t) = A_t^R - A_t^C$, A_t^R is the weighted average age of pensioners and A_t^C is the weighted average age of contributors. The turnover duration can be interpreted as the time expected to elapse from the moment that one monetary unit enters the system as a contribution until it leaves as a pension assuming that economic, demographic and legal conditions remain constant, that is, the time in years that is expected to elapse before all system liabilities are renewed or rotated¹⁰.

The Liabilities

Under a NDC framework, liabilities are calculated using the retrospective method, whose value coincides with the current cohort's notional capital. The expected value of the liabilities using the prospective method produces the same results only if the adjustment A_t is used (see Appendix B).

The sum of the liabilities of all participants is denoted by $V(t)$, where liabilities towards contributors at time t during periods 1 and 2 are denoted by $V_C^i(t)$ for $i = 1, 2$ and liabilities towards retirees at time t during periods 3 and 4 are denoted by $V_R^i(t)$ as follows:

$$\begin{aligned}
 V_C^1(t) &= NDC(y, t) = 0 \\
 V_C^2(t) &= NDC(y + 1, t) = \pi S(y, t - 1) I(y, t - 1) I(t) = C(t) \frac{S(y, 0) e^{R_{t-1}}}{K_C(t - 1)} \\
 V_R^3(t) &= NDC(y + 2, t) = C(t) K_N(t) \\
 V_R^4(t) &= NDC(y + 3, t) = (NDC(y + 2, t - 1) - P(y + 2, t - 1)) I(t) = C(t) K_N(t - 1) \frac{a_{t-1} - 1}{a_{t-1}} \\
 V(t) &= V_C^1(t) + V_C^2(t) + V_R^3(t) + V_R^4(t) = C(t) K_V(t)
 \end{aligned} \tag{16}$$

where

$K_V(t) = 2 + \frac{S(y, 0) e^{R_{t-3}}}{K_C(t - 3)} - \frac{K_N(t - 1)}{a_{t-1}}$. The liabilities are proportional to the current contributions of the active population.

The following result demonstrates that this ratio is generally not equal to 1. A solvency ratio greater than 1 does not directly imply a liquidity ratio greater than 1. Moreover, a balance ratio less than 1 does not imply system deficit.

Proposition 2. *In general, contribution assets do not equal liabilities in this four-period overlapping generations unfunded dynamic model, i.e., $CA(t) \neq V(t) \forall t$.*

Proof. This result is proven by counterexample. The contribution assets equal liabilities if and only if $TD(t) = K_V(t) \forall t$. Assume that $R_t = R = \text{cte}$ and $\gamma_t = \gamma = \text{cte} \forall t$ and deterministic. Thus,

¹⁰For more details, see Boado-Penas et al. (2008) and Settergren & Mikula (2005).

$$TD(t) = 2 + \frac{1}{K_O(t)} - \frac{S(y+1,0)}{S(y,0)e^R + S(y+1,0)} - \frac{1}{K_O(t)} \frac{1}{a_{t-1}}$$

$$K_V(t) = 2 + \frac{S(y,0)}{S(y,0)e^R + S(y+1,0)} - \frac{1}{a_{t-1}}$$

These two expressions are equal if and only if $K_O(t) = 1$, which only occurs when the system is in the steady state. $\square$

Design of an Automatic Balance Mechanism (ABM)

As demonstrated in the previous section, a notional defined contribution pension system does not guarantee liquidity or solvency by design in a dynamic environment. The desired liquidity and solvency objectives can be reached through automatic balancing mechanisms (ABMs). According to Vidal-Meliá et al. (2010), an automatic balance mechanism is a set of predetermined measures established by law to be applied immediately as required according to an indicator that reflects the financial health of the system. The purpose of successive application of AMBs is to provide automatic financial stability, which can be defined as ‘the capacity of a pension system to adapt to financial turbulence without legislative intervention’ (Settergren (2013)). It is understood that turbulence can be caused by economic, financial or demographic shocks that affect the system’s financial equilibrium. These ABMs are used to depoliticize the management of pay-as-you-go systems by adopting measures suited to long-term planning.

The following questions arise in designing an Automatic Balance Mechanism:

- Which type of automatic balance mechanism should be applied?
- Should an automatic balance mechanism be symmetric or asymmetric?

This section will focus on two types of automatic balance mechanisms. The first mechanism restores liquidity to the system, whereas the second mechanism reestablishes solvency as defined in the previous section. Both mechanisms will be considered for symmetric and asymmetric cases. A symmetric automatic balance mechanism affects pension system participants during positive and negative economic scenarios, while an asymmetric automatic balance mechanism only affects participants during negative changes, allowing surpluses to accumulate under positive economic scenarios. Although symmetric automatic balance mechanisms are the fairest in fully sharing positive and negative changes, they are not usually applied¹¹.

¹¹In practice, the Swedish ABM allows for recovery. After a period of low returns as a consequence of the mechanism, a period of higher-than-normal returns follows. For more detail explanations, see Appendix C and Barr and Diamond (2011).

Asymmetric automatic balance mechanisms can increase the value of buffer funds, which attenuate unexpected negative shocks to the system and smooth the impact on the insured¹².

ABMs adjust the notional factor through a time-dependent variable to obtain liquidity or solvency ratios of at least unity at all times. The adjusted notional factor applied to the system to attain the desired equilibrium will be $I_z(t) = I(t)B_z(t)$, where $z = LR$ when the automatic balance mechanism for liquidity equilibrium is used and $z = SR$ when the automatic balance mechanism for solvency equilibrium is used. ABMs affect contributors by adapting the notional factor by which their notional capital is accumulated and retirees through changes in pension indexation.

Formulas (7), (12), (4), (15) and (16) correspond to formulas (17), (18), (19), (20) and (21) in the presence of an automatic balance mechanism with $z = LR, SR$. In formula (5), the contribution revenue at time t remains the same while pension expenditures change through indexation that also affects the value of the fund.

$$NDC^z(y+2, t) = C(t)B_z(t)K_N^z(t) \quad (17)$$

$$O^z(t) = C(t)B_z(t)K_O^z(t) \quad (18)$$

$$F_z^-(t) = F_z(t-1)^+ e^{r_t} = F(0)e^{\sum_{i=1}^t r_i} + \sum_{j=0}^{t-1} (C(j) - O^z(j))e^{\sum_{i=j+1}^t r_i} \quad (19)$$

$$CA^z(t) = C(t)TD^z(t) \quad (20)$$

$$V^z(t) = C(t)B_z(t)K_V^z(t) \quad (21)$$

where

$$K_N^z(t) = \frac{S(y, 0)e^{R_{t-2}}}{K_C(t-2)} B_z(t-1) + \frac{S(y+1, 0)}{K_C(t-1)} \quad (22)$$

$$K_O^z(t) = \frac{K_N^z(t)}{a_t} + B_z(t-1)K_N^z(t-1)\frac{a_{t-1}-1}{a_{t-1}} \quad (23)$$

$$K_V^z(t) = 1 + B_z(t-1) \left\{ \frac{S(y, 0)e^{R_{t-2}}}{K_C(t-2)} + K_N^z(t-1)\frac{a_{t-1}-1}{a_{t-1}} \right\} \quad (24)$$

$$TD^z(t) = 2 + B_z(t-1)\frac{K_N^z(t-1)}{K_O^z(t)}\frac{a_{t-1}-1}{a_{t-1}} - \frac{S(y+1, 0)}{K_C(t)} \quad (25)$$

The automatic balance mechanism for the liquidity ratio

This automatic balance mechanism is designed to attain a liquidity ratio of 1. The notional factor is therefore affected by $B_{LR}(t)$; therefore, $I_{LR}(t) = I(t)B_{LR}(t)$. The liquidity ratio that incorporated the automatic mechanism can be written as follows:

¹²See Kaas et al. (2009) for the global definition of risk aversion and Settergren (2001) for its link to notional defined contribution pension systems.

$$B_{LR}(t) = \frac{C(t) + F_{LR}^-(t)}{C(t)K_O^{LR}(t)} \quad (26)$$

If the automatic balance mechanism is symmetric and the initial fund is equal to 0 when the automatic balance mechanism is first applied, the expression is reduced to $B_{LR}(t) = \frac{1}{K_O^{LR}(t)}$ because no funds can be accumulated. In this case, the automatic balance mechanism behaves as a pure defined contribution system, providing participants with available funds in both good and bad times.

If the automatic balance mechanism is asymmetric, participants do not benefit from positive economic shocks and funds can accumulate according to formula (19) for $z=LR$.

The automatic balance mechanism for the solvency ratio

This automatic balance mechanism is designed to restore the system solvency. The adjusted notional factor is then $I_{SR}(t) = I(t)B_{SR}(t)$. The Contribution Asset at time t does not depend on the automatic balance mechanism $B_{SR}(t)$ at time t , as noted in formula (20) and (25). Only the liabilities, $V^{SR}(t)$, depend on the current automatic balance mechanism factor. The solvency ratio that includes this mechanism can be represented as follows:

$$B_{SR}(t) = \frac{CA^{SR}(t) + F_{SR}^-(t)}{V^{SR}(t)} = \frac{TD^{SR}(t) + \frac{F_{SR}^-(t)}{C(t)}}{K_V^{SR}(t)} \quad (27)$$

This automatic balance mechanism might accumulate some funds because a solvency ratio equal to or greater than 1 does not imply a liquidity equal to or greater than 1. This ABM essentially reduces the growth of pension liability, that is, pension payments and contributor notional pension capital.

Numerical illustration

This section presents a numerical example and analyses the behavior of the notional factor under both ABMs in a stochastic environment. The effects of these automatic balance mechanisms are evaluated in terms of the expected value and variance of the notional factor and the expected value of the ratio of the buffer fund and contributions¹³.

¹³The ratio provides more insights than examining the nominal value of the fund because we can easily compare the values of contributions.

The entry population process for $x = y$ and salary process for $x = y, y + 1$ follow a geometric Brownian motion. The processes are described as follows:

$$dl(y, s) = R l(y, s)ds + \sigma_R l(y, s)dw_R(s) \quad (28)$$

$$dS(x, s) = \gamma S(x, s)ds + \sigma_\gamma S(x, s)dw_\gamma(s) \quad (29)$$

The continuous processes (28) and (29) generate discrete processes (30) and (31) when $s = t$. The change in the rates of the entry population and salaries at time t are represented as follows:

$$D_t = \frac{l(y, t)}{l(y, t-1)} = e^{Rt} = e^{R - \frac{\sigma_R^2}{2} + \sigma_R(w_R(t) - w_R(t-1))} \quad (30)$$

$$S_t = \frac{S(x, t)}{S(x, t-1)} = e^{\gamma t} = e^{\gamma - \frac{\sigma_\gamma^2}{2} + \sigma_\gamma(w_\gamma(t) - w_\gamma(t-1))} \quad (31)$$

The stochastic process D_t (resp. S_t) is distributed log-normally with mean $R - \sigma_R^2/2$ and standard deviation σ_R (resp. with mean $\gamma - \sigma_\gamma^2/2$ and standard deviation σ_γ) $\forall t$. In this context, the notional factor (6) is represented as follows:

$$I(t) = S_t D_{t-1} \frac{S(y, 0)D_t + S(y + 1, 0)}{S(y, 0)D_{t-1} + S(y + 1, 0)} \quad (32)$$

The demographic and the salary processes, D_t and S_t , respectively, are stationary stochastic processes, which means that their distributions do not depend on time. This holds in the absence of exogenous shocks, such as a drop in fertility or a baby boom. For instance, if the demographic process at time t is affected by a one-time exogenous shock δ , the demographic process at time t is represented by $D_t^* = D_t e^\delta$.

The processes w_R and w_γ are correlated wiener processes, that is, the expected value of $w_R(s)w_\gamma(s)$ is equal to ρ_s . Market returns are considered independent of the wage and population processes although they might be correlated¹⁴. The automatic balance mechanism is applied at time t for the first time. Before this moment t , no funds have accumulated. The details of the theoretical framework and recurring formulas used for the calculations as well as further details concerning the joint distribution of the random vectors are presented in Appendix D.

To calculate the expected value and variance of the notional factor and the expected value of the ratio of the buffer fund to the contribution, we use the closed-form expressions developed above in 'A dynamic four overlapping generations model' and the tools described in Appendix D.

We explore these results under symmetric and asymmetric designs. In the symmetric case, the notional factor increases when the system is liquid and/or solvent and

¹⁴For more details, see Cannon and Tonks (2013) and McDonald and Cairns (2009).

decreases during periods of deficit and/or insolvency. In the asymmetric design, changes in the notional factor occur only during deficit and/or insolvency.

The following are main assumptions of our numerical model:

- Two generations of contributors and two generations of pensioners coexist at each moment in time.
- At time $t=0$, the contribution base is 30,000 for individuals aged y , and 45,000 for those aged $y+1$.
- The entry population process follows a geometric Brownian motion, where the percentage drift and volatility R and σ_R equal 0.25% and 5%, respectively.
- The salary process follows a geometric Brownian motion, where the percentage drift and volatility γ and σ_γ equal 1.5% and 10%, respectively.
- The wiener processes for salaries and population are correlated with $\rho = -0.25$. The negative correlation reflects the fact that cohort size negatively affects wages, especially for highly educated individuals (see Brunello, 2010).
- The return of the buffer fund r_t is 0% for all periods.
- The probability of dying equals 0.5 for the first generation of pensioners and 1 for the second generation. There is no mortality before retirement. Three different scenarios (base, up and down) for mortality are considered. First, we consider constant mortality rates (base scenario). Second, the survival probability is assumed to increase 0.005 per period (up scenario). Third, the survival probability is assumed to decrease 0.005 over time (down scenario).

The results are obtained through Monte Carlo simulations for 1 million randomly generated paths. The analysis is conducted for 8 periods, which represents two full population renewals.

This paper first presents the numerical results when future longevity patterns are known (Case 1) and then addresses mortality risk.

In the first line of Figure 2(a), we observe that the value of the expected notional factor under the liquidity ratio automatic balance mechanism is higher than under the solvency ratio automatic balance mechanism. The solvency ratio automatic balance mechanism is not triggered in this scenario; therefore, the notional factor under a non-automatic balance mechanism provides the same results.

However, in a scenario of increasing longevity, the second line of Figure 2, the expected notional factor is higher in the presence of either automatic balance mechanism. This result is expected because the longevity trend is known in advance and the initial pension is reduced to adapt to the new mortality pattern. Consequently, the expected value of the buffer fund with automatic balance mechanisms, Figure 4, is always lower than the non-automatic balance mechanism due to the increase in the notional factor after applying an automatic balance mechanism. The opposite pattern is observed with decreasing longevity trend, where the automatic balance

Figure 2: The expected value of the notional factor after the introduction of an automatic balance mechanism: without ABM (dotted black line), liquidity ratio ABM (dark gray squares) and solvency ratio ABM (light gray rhombuses)

Note: The first row of the graphic corresponds to the base longevity scenario, the second row corresponds to the up longevity scenario, and the third row corresponds to the down longevity scenario. *Source:* the authors.

mechanisms decrease the expected notional factor while maintaining the value of the buffer fund near zero.

Under an asymmetric design (Figure 2), it is not surprising that the activation of an automatic balance mechanism reduces the expected value of the notional factor in all longevity scenarios because the asymmetric automatic balance mechanisms are designed to be capped at 1. When we calculate the average value of the million scenarios, values less than 1 will have higher relative weights, which significantly decrease the expected value of the notional factor. This reduction is greater for the solvency ratio automatic balance mechanism. In the symmetric design, this automatic balance mechanism also has the lowest notional factor variance (Figure 3 and Table 2).

In all scenarios, the expected value of the notional factor under the liquidity ratio automatic balance mechanism and variance is higher than under a solvency ratio automatic balance mechanism (Figure 3). In all scenarios, the solvency ratio automatic balance mechanism produces the lowest values of the expected notional factor and its variance, which are depicted in Table 2.

The average, 75, 95 and 99 percentiles¹⁵ over the entire study period indicate that the preferred automatic balance mechanism is the liquidity ratio when the mechanism is symmetric and no automatic balance mechanism when the mechanism is asym-

¹⁵The numerical results associated with the percentiles are not depicted to limit the length of the paper.

Figure 3: Variance of the notional factor after the introduction of an automatic balance mechanism: without ABM (dotted black line), liquidity ratio ABM (dark gray squares) and solvency ratio ABM (light gray rhombuses)

Note: The first row of the graphic corresponds to the base longevity scenario, the second row corresponds to the up scenario, and the third row corresponds to the down longevity scenario. *Source:* the authors.

metric. This result is consistent with the expected values provided Figure 2. In the asymmetric case, the highest expected value occurs with the absence of an automatic balance mechanism.

Mortality risk

The system is exposed to mortality risk to the extent that we do not know the survival probability of individuals. It is well understood that this risk has an impact on the financial health of pension systems. Hence, Case 2, in which the current historical values of mortality are used, is examined to evaluate to what extent mortality risk affects our results. In this case, mortality rates are based on verifiable facts; therefore, current longevity is used even though it is expected to change. Note that in Case 2, we no longer have a fair pension system. Furthermore, we assume that the indexation process $\lambda(t)$ is equal to the discount factor, which is equal to the notional factor $I_z(t)$ for $z = LR, BR$. Therefore, the annuity for Case 1 is $a_t^1 = \mathbb{E}_t [1 + p_{t+1}]$, whereas the annuity in the second case is $a_t^2 = \mathbb{E}_t [1 + p_t]$.

As indicated in Figures 2, 3 and 4 the results vary if current mortality rates are used (Case 2) rather than prospective mortality rates (Case 1). In both symmetric and non-symmetric designs, under increasing longevity, the value of the expected notional factor is lower after applying an automatic balance mechanism because the automatic balance mechanisms correct errors introduced by using current mortality rates, which resulted in higher initial pensions. Note that the notional factor decreases further using a non-symmetric automatic balance mechanism. The con-

Figure 4: Expected value of the ratio between the fund and contributions after the introduction of an Automatic Balance Mechanism: without ABM (dotted black line), liquidity ratio ABM (dark gray squares) and solvency ratio ABM (light gray rhombuses)

Note: The first row of the graphic corresponds to the base longevity scenario, the second row corresponds to the up longevity scenario and the third row corresponds to the down longevity scenario. *Source:* the authors.

trary occurs in Figure 2 under decreasing longevity for a symmetric mechanism. For the asymmetric case, the explanation is the same as that for Case 1 for the up scenario. The automatic balance mechanisms are capped at 1, which produces a lower global average by assigning lower values higher relative weights. As in case 1, the expected value of the notional factor and its variance under the liquidity ratio automatic balance mechanism is higher than under the solvency ratio automatic balance mechanism for both symmetric and asymmetric designs (Table 2).

The effect of mortality on the expected value of the fund is straightforward for the symmetric case without automatic balancing mechanisms, depicted in Figure 4. The expected value in the Up (Down) scenario enters a state of systematic debt (surplus), whereas with the introduction of an automatic balance mechanism, the expected value of the fund approaches zero. In Case 1, the opposite pattern can be clearly observed through the expected value of the buffer fund depicted in Figure 4.

In the asymmetric case, automatic balance mechanisms increase the expected value of the fund. If longevity increases but current mortality rates are used in the pension calculation, the amount of the initial pension increases (see Figure 4). This results in higher pension expenditures and a corresponding decrease in the value of the fund. The automatic balance mechanisms consequently decrease the notional factor to establish liquidity or solvency. Finally, we observe that asymmetric mechanisms for Cases 1 and 2, the expected value of the fund increases more than that of its symmetric counterpart because the much lower average notional factor systematically

Figure 5: Expected Value of the notional factor and the ratio between the fund and contributions after a baby boom with an Automatic Balance Mechanism: no ABM (dotted black line), liquidity ratio ABM (dark gray squares) and solvency ratio ABM (light gray rhombuses)

Note: The first row of graphics corresponds to the base longevity scenario, the second row corresponds to the up longevity scenario and the third row corresponds to the down longevity scenario. *Source:* the authors.

increases the size of the buffer fund. Note that the final values of the expected value of the fund in the asymmetric design approach 2, which means that the value of the fund is two times greater than the value of contributions at the same time. This value implies that asymmetric automatic balance mechanism may be too conservative and lead to excessive capital accumulation under our assumptions.

Baby boom

To evaluate a demographic shock, we examine the impact of one-time positive exogenous shock that affects the entrant population at time t . This shock is denoted by δ and is assumed to have a value of 10%. The effects of this shock are illustrated by a so-called baby boom.

This model is based on a stationary stochastic process, which means that one-time shocks can only occur exogenously. We included a positive shock, which replicates the effect of a fertility boost as indicated by the phenomenon's name, baby boom. We evaluate the financial health of the pension system after a baby boom and observe the automatic balance mechanism reaction. Figure 5 indicates the expected value of the notional factor and the ratio of the fund to the contribution level for the symmetric design¹⁶.

We observe that the notional factor increases substantially from the average value

¹⁶The asymmetric case results are omitted because the explanation is similar to the non-baby boom case .

of approximately 1.018 due to the sudden entrant population increase, which increases the total contribution base substantially. Once the baby boom generation retires, there is no longer an effect on the notional factor because the entry population returns to the relative stationary level after the exogenous shock. Note that the notional factor in absence of automatic balance mechanism does not depend on the retired population but on the contributors, which explains that the level of the notional factor goes back to the pre-shock level once that the shocked generation retires.

The liquidity ratio balance mechanism increases the notional rate during the first two periods due to the surplus created by the higher relative contribution level. However, once the baby boom generation retires, this factor decreases significantly to avoid debt.

The solvency ratio balance mechanism responds differently to this shock due to its construction. When the baby boom generation enters the pension system, turnover duration increases because it assumes that the future population will have the same composition, that is, it behaves as if the current shock is permanent. This anticipation increases the level of the notional factor affected by the balance ratio during the first period, which affects both contributors and retirees. This first leads to debt as the expenditure increase above its sustainable levels. This increase affects the solvency balance mechanism during the second period, which decreases the notional factor. The decrease is sufficient to restore a surplus level, which will positively affect the notional factor. The notional rate affected by both balance mechanisms approaches the level in absence of adjustments once the shock generation exits the pension system.

In absence of automatic balance mechanisms, the system only partially regulates itself after the shock population exits the system, as indicated by the triangle in Figure 5. For instance, in Case 1 of the longevity base scenario, we observe that the triangle occurs because pension expenditures equal contributions as observed in the expression (12). The pre-contribution fund equals 0 during the second period because we assume an initial fund of 0. The level of the notional rate, although higher than its normal levels, is insufficient to attain fund neutrality. Therefore, a surplus occurs. However, this surplus will be used to pay the pensions to the higher number of baby boom pensioners until they leave the pension system, decreasing the fund to a level approaching but not reaching zero. This interpretation is independent of the market interest rate and is due to the simple nature of the shock.

Finally, our results are robust to alternative assumptions¹⁷. For simplicity, we provide only some of the representative results.

¹⁷We also addressed the possibility that contributors aged $y + 1$ earn less than contributors aged y do. Furthermore, we considered the possibility of a negative base notional rate with a negative sum of γ and R . Finally, we examined a case with negative parameter R as well as a case with a negative exogenous shock δ .

Table 2: The variance of the Notional Factor: Aggregated values

	NO BABY BOOM						BABY BOOM					
	Case 1			Case 2			Case 1			Case 2		
	Base	Up	Down	Base	Up	Down	Base	Up	Down	Base	Up	Down
	SYMMETRIC											
No ABM	0,08578	0,08578	0,08578	0,08578	0,08578	0,08578	0,08843	0,08843	0,08843	0,08843	0,08843	0,08843
LR	0,09102	0,09113	0,09089	0,09102	0,09078	0,09126	0,09391	0,09403	0,09378	0,09391	0,09367	0,09416
BR	0,08408	0,08411	0,08404	0,08408	0,08404	0,08410	0,08674	0,08678	0,08670	0,08674	0,08670	0,08677
	NO SYMMETRIC											
No ABM	0,08578	0,08578	0,08578	0,08578	0,08578	0,08578	0,08843	0,08843	0,08843	0,08843	0,08843	0,08843
LR	0,08630	0,08609	0,08661	0,08630	0,08644	0,08619	0,08890	0,08869	0,08920	0,08890	0,08903	0,08879
BR	0,08529	0,08545	0,08509	0,08529	0,08521	0,08536	0,08776	0,08792	0,08757	0,08776	0,08768	0,08784

Table 3: The variance of the ratio between the fund and contributions: Aggregated values

	NO BABY BOOM				BABY BOOM							
	Case 1		Case 2		Case 1			Case 2				
	Base	Up	Down	Base	Up	Down	Base	Up	Down	Base	Up	Down
	SYMMETRIC											
No ABM	0,00106	0,00104	0,00111	0,00106	0,00107	0,00105	0,00106	0,00104	0,00111	0,00106	0,00107	0,00105
LR	0,00000	0,00000	0,00000	0,00000	0,00000	0,00000	0,00065	0,00076	0,00052	0,00065	0,00058	0,00072
BR	0,00057	0,00057	0,00057	0,00057	0,00057	0,00057	0,00059	0,00068	0,00052	0,00059	0,00057	0,00063
	NO SYMMETRIC											
No ABM	0,00102	0,00101	0,00106	0,00102	0,00103	0,00102	0,00102	0,00101	0,00106	0,00102	0,00103	0,00102
LR	0,00000	0,00000	0,00000	0,00000	0,00000	0,00000	0,00070	0,00083	0,00058	0,00070	0,00063	0,00078
BR	0,00056	0,00057	0,00056	0,00056	0,00056	0,00056	0,00073	0,00080	0,00066	0,00073	0,00068	0,00077

The tables depict the sum of the variances calculated at different moments of time during the whole period studied. *Source:* the authors.

Conclusion

This paper describes how liquidity and solvency indicators in notional defined contribution accounts are affected by fluctuations in economic and demographic conditions in a stochastic environment. The analysis uses a four-generation model in which two generations are contributors and two generations are pensioners. The consideration of four cohorts introduces heterogeneity to the pension system while maintaining a tractable dependence structure. Furthermore, we can draw interesting conclusions without the drawback of heavier computation.

Under this scenario, the notional factor that indexes the accumulated notional capital is affected by both salary and population processes. The pension is revalued by an adjusted indexation process that guarantees longitudinal equilibrium among the cohorts. However, this equilibrium, also referred as actuarial fairness, does not guarantee liquidity or solvency of the pension system in a dynamic framework. Consequently, some automatic balancing mechanism should be triggered.

An automatic balancing mechanism is a predetermined measure established by law to be applied immediately according to an indicator, i.e., solvency or liquidity. The purpose of an ABM is to allow the pension system to adapt to financial turbulence without legislative intervention. This paper considers two automatic balancing mechanisms. The first mechanism makes income from contributions equal pension expenditures (liquidity ratio automatic balance mechanism), and the second mechanism makes the assets of the pension system equal liabilities (solvency ratio automatic balance mechanism). The results are evaluated in terms of the expected values and variances of the notional factor and buffer fund and are calculated under both symmetric and asymmetric designs. In most of cases, the introduction of an ABM reduces the volatility of the fund. The solvency ratio automatic balance mechanism yields the lowest value of the notional factor's variance and the lowest expected value.

The paper also indicates how longevity risk affects our analysis. It might be beneficial to consider longevity trends to calculate annuities, especially when the trend is increasing longevity. This adjustment would avoid debt accumulation in absence of automatic balance mechanisms. Furthermore, note that asymmetric automatic balance mechanisms can lead to significant capital accumulation after two population renewals. This is consistent with the fact that an 'NDC scheme does little to prevent significant asset accumulation (...) on average' as argued by Auerbach and Lee (2009).

These conclusions are robust to a one-time exogenous demographic shock. Additionally, the notional system that uses the rate of increase of the contribution base as the compounding factor can regulate itself almost completely in presence of such a shock, that is, once the shock generation exists the pension system, the level of debt or surplus approaches the level expected in absence of a shock. The choice of automatic mechanism depends on the preferences of legislators in terms of expected value, variance of the notional rate and level of the buffer fund.

Finally, based on the two ABMs presented in this paper, at least three important

directions for future research can be identified:

- To identify the ABM preferences of pension system participants and legislators as well as of contributors and pensioners. According to Barr & Diamond (2011), the current Swedish balancing mechanism has the undesirable consequence of favoring workers over retirees.
- To present an automatic balancing mechanism that smooths volatility over the period and partially transfers to future system participants, and to compare participant and legislator preferences towards this ABM.
- To extend the four overlapping generations model to a real society and assess the impact of economic, financial and demographic conditions on the automatic balancing mechanism.

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Appendix

Appendix A: Derivation of the adjustment factor A_t

In practice, model parameters are rarely consistent with expectations. However, assumptions must be made about the future to calculate annuities. Problems arise when realized values differ from the expected values. In this case, longitudinal equilibrium is never reached, which means that plan members receive less or more than their contributions to the system. Prospective liabilities (calculated by considering the present value of future cash flows) are not equal to retrospective liabilities (calculated by considering the compounded value of past cash flows).

The appendix demonstrates that that adjustments can be made to attain this longitudinal equilibrium at any time. However, this result implies that indexation of pensions in practice would differ among retirees cohorts, which is not a desirable political solution. These adjustments thus only produce actuarial fairness. There is a difference between the theoretical processes set by the legislator or pension reg-

ulator and the observed values of the processes. The processes $i(t)$, m_t and $\lambda(t)$ set by the legislator for the period t denote, respectively, the theoretical notional factor, mortality rate and indexation of pensions. The processes $I_Z(t)$, p_t and $\Lambda(t)$ denote, respectively, the observed notional factor, which could be affected by some automatic balance mechanism $B_Z(t)$, mortality rate and indexation rate at time t . In this framework, the annuity is represented as follows:

$$a_t = \mathbb{E}_t \left[1 + m_{t+1} \frac{\lambda(t+1)}{i(t+1)} \right] \quad (33)$$

The aim is to produce equivalence between retrospective and prospective liabilities for all cohorts, in particular, for the cohort aged $y+2$ at time t . Retrospective liability is denoted by (34) and prospective liability is denoted by (35). The adjustment factor at time $t+1$ by A_{t+1} is represented as follows:

$$L^R(y+2, t) = P(y+2, t)l(y+2, t) + P(y+2, t)\Lambda(t+1)A_{t+1}l(y+3, t+1)\frac{1}{I_Z(t+1)} \quad (34)$$

$$L^P(y+2, t) = NDC(y+2, t)\frac{1}{a_t} \left(1 + p_{t+1} \frac{\Lambda(t+1)A_{t+1}}{I_Z(t+1)} \right) \quad (35)$$

The following adjustment factor makes both liabilities equivalent:

$$A_{t+1} = \frac{I_Z(t+1)}{\Lambda(t+1)} \frac{1}{p_{t+1}} \mathbb{E}_t \left[m_{t+1} \frac{\lambda(t+1)}{i(t+1)} \right] \quad (36)$$

If the independence of mortality, demographic and economic processes is assumed, the adjustment A_{t+1} can be expressed as follows:

$$A_{t+1} = \underbrace{\frac{I_Z(t+1)}{\Lambda(t+1)} \mathbb{E}_t \left[\frac{\lambda(t+1)}{i(t+1)} \right]}_{\text{Economic adjustment}} \underbrace{\frac{\mathbb{E}_t[m_{t+1}]}{p_{t+1}}}_{\text{Longevity adjustment}} \quad (37)$$

The economic adjustment affects all retirees because the calculation only considers the difference between theoretical and observed parameters. However, the second part, which corresponds to the longevity adjustment, would differ by retiree cohort because it depends on the assumed mortality at the time of annuity calculation and the actual mortality of that cohort at retirement. In Sweden, the annuity that is used in the calculation of the first pension considers an interest rate of 1.6%, which increases the amount of the initial pension. However, to maintain financial stability, pension adjustments must be made when real economic growth deviates from 1.6%. See the Orange Report from the Swedish Pension Agency (2013) for further details.

Appendix B: Derivation of the retrospective and prospective liabilities

In this section, the retrospective and prospective liabilities are calculated for each cohort at time t . The prospective (retrospective) liabilities for the cohort aged x at time t will be denoted by $L^P(x, t)$ ($L^R(x, t)$). These liabilities are calculated prior to any payment.

The following are the liabilities for the contributing cohort aged y :

$$L^P(y, t) = 0 \times l(y, t) = 0 \quad (38)$$

$$\begin{aligned} L^R(y, t) &= \frac{P(y+2, t+2)l(y+2, t+2)}{I_Z(t+1)I_Z(t+2)} + \frac{P(y+2, t+2)\Lambda(t+3)A_{t+3}p_{t+3}l(y+2, t+2)}{I_Z(t+1)I_Z(t+2)I_Z(t+3)} \\ &\quad - \pi S(y, t)l(y, t) - \frac{\pi S(y+1, t+1)l(y+1, t+1)}{I_Z(t+1)} \\ &= \frac{NDC(y+2, t+2)}{I_Z(t+1)I_Z(t+2)} \frac{1 + p_{t+3} \frac{\Lambda(t+3)A_{t+3}}{I_Z(t+3)}}{a_{t+2}} - \frac{NDC(y+2, t+2)}{I_Z(t+1)I_Z(t+2)} = 0 \end{aligned} \quad (39)$$

The following are liabilities for the contributing cohort aged $y+1$:

$$L^P(y+1, t) = \pi S(y, t-1)l(y, t-1)I_Z(t) \quad (40)$$

$$\begin{aligned} L^R(y+1, t) &= \frac{P(y+2, t+1)l(y+2, t+1)}{I_Z(t+1)} + \frac{P(y+2, t+1)\Lambda(t+2)A_{t+2}p_{t+2}l(y+2, t+1)}{I_Z(t+1)I_Z(t+2)} \\ &\quad - \pi S(y+1, t)l(y+1, t) \\ &= \frac{NDC(y+2, t+1)}{I_Z(t+1)} \frac{1 + p_{t+2} \frac{\Lambda(t+2)A_{t+2}}{I_Z(t+2)}}{a_{t+1}} - \pi S(y+1, t)l(y+1, t) \\ &= \left(\pi S(y, t-1)l(y, t-1)I_Z(t) + \pi S(y+1, t)l(y+1, t) \right) \frac{I_Z(t+1)}{I_Z(t+1)} \\ &\quad - \pi S(y+1, t)l(y+1, t) \\ &= \pi S(y, t-1)l(y, t-1)I_Z(t) \end{aligned} \quad (41)$$

The following are liabilities for the retired cohort aged $y+2$:

$$L^P(y+2, t) = NDC(y+2, t) \quad (42)$$

$$\begin{aligned} L^R(y+2, t) &= P(y+2, t)l(y+2, t) + P(y+2, t)\Lambda(t+1)l(y+3, t+1) \frac{1}{I_Z(t+1)} \\ &= NDC(y+2, t) \frac{1 + p_{t+1} \frac{\Lambda(t+1)A_{t+1}}{I_Z(t+1)}}{a_t} = NDC(y+2, t) \end{aligned} \quad (43)$$

Finally, the following are the liabilities for the retired cohort aged $y+3$:

$$\begin{aligned}
L^P(y+1, t) &= (NDC(y+2, t-1) - P(y+2, t-1)l(y+2, t-1)) I_z(t) \\
&= NDC(y+2, t-1) \frac{a_{t-1} - 1}{a_{t-1}}
\end{aligned} \tag{44}$$

$$\begin{aligned}
L^R(y+1, t) &= P(y+3, t)l(y+3, t) = P(y+2, t-1)\Lambda(t)A_t p_t l(y+2, t-1) \\
&= NDC(y+2, t-1) \frac{a_{t-1} - 1}{a_{t-1}}
\end{aligned} \tag{45}$$

Appendix C: Aim and purpose of the Swedish Solvency Ratio

The solvency ratio used in Sweden has two purposes: measuring whether the system can fulfil its obligations to its contributors and deciding whether the automatic balance mechanism should be applied.

Following Settergren (2001), if the solvency ratio falls below one, the automatic balance mechanism is triggered as shown in Figure 6. This essentially reduces the growth of pension liability, i.e., pension payments and contributor notional pension capital. Thus, the balance index rather than the change in average salary (expressed by an income or salary index) is used to revalue the pensions and notional account of each contributor.

Figure 6: Income versus Balance Index

Source: Settergren (2001)

The following expression is used to calculate the balance index in year t , the first year in a period when the solvency ratio is less than one:

$$BI_t = \frac{I_{t+i}}{I_{t+i-1}} SR_t \tag{46}$$

where BI_t is the balance index in year t ; I_t is the income index in year t , an index for the average income in year t ; and SR_t is the solvency ratio in year t . For year $t+i$, the following represents the balance index:

$$BI_{t+i} = \frac{I_{t+i}}{I_{t+i-1}} SR_{t+i} BI_{t+i-1} = I_{t+i} \prod SR_{t+i} \quad (47)$$

where BI_{t+i} is the balance index in year $t+i$; I_{t+i} is the income index for year $t+i$, which expresses the average accumulated variation in salaries up to year $t+i$; and SR_t is the solvency ratio in year $t+i$.

If the solvency ratio is greater than one when the mechanism is triggered, the revaluation of the contributors' notional pension capital and pensions in payment will be greater than the change in average salary (see Figure 6). This adjustment will continue until the balance index reaches the level of the income index. This means that the pensions paid during the whole balance period will reach the value they would have reached if the mechanism had not been activated.

This procedure for calculating the balance index is repeated successively until year s in which the mechanism is deactivated because the value of the balance index is equal to or greater than that of the income index. After year s , the mechanism is deactivated and variation in pensions and notional accounts equal the variation in average salary.

In this paper, two automatic balance mechanisms are presented for the solvency ratio, one symmetric and one asymmetric. Although the asymmetric automatic balance mechanism shares many characteristics of the Swedish mechanism, it differs in its implementation. In our case, the automatic balance mechanism is not allowed to exceed 1 even after triggering the automatic balancing system, whereas the Swedish mechanism can exceed one until the balance index equals the income index as follows:

$$BI_t = \frac{C_{t+i}}{C_{t+i-1}} \text{Min}[SR_t, 1] \quad (48)$$

where BI_t is the balance index in year t ; C_t is the income from contributions in year t ; and SR_t is the solvency ratio in year t .

Appendix D: Theoretical framework

The expressions of the automatic balance mechanisms (26) and (27) do not allow specification of their probability distributions. Therefore, the expected value calculation cannot occur through analytical formulas. However, given two random variables X and Y and a function $g(X, Y)$, the expected value of this function can be expressed directly in terms of the transformation function $g(X, Y)$ and the joint density $f_{X,Y}(x, y)$ of X and Y : $E[g(X, Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f_{X,Y}(x, y) dx dy$ (see Papoulis, 1992). Furthermore, if the two random variables X and Y are independent, the joint density is $f_{X,Y}(x, y) = f_X(x) f_Y(y)$. This property can be applied to our problem. We will present the recursive formulas for our framework.

Denote the transformation of the random variable X_t at time $s > t$ for the automatic balance mechanism $z=LR,SR$ as $g_z(t, s, x_s)$. Note that t indicates that the automatic balance mechanism is first implemented at time t .

$$TD^Z(s) = \left(\begin{array}{c} 2 + \frac{a_{s-1}-1}{a_{s-1}} \left[\frac{S(y,0)D_{s-3}}{K_C(s-3)} g_z(t, s-1, x_{s-2}) + \frac{S(y+1,0)}{K_C(s-2)} \right] \frac{g_z(t, s-1, x_{s-1})}{K_O(s)} \\ - \frac{S(y+1,0)}{K_C(s)} \end{array} \right) \\ = f_{TD^Z(s)}(D_s, D_{s-1}, D_{s-2}, D_{s-3}, g_z(t, s-1, x_{s-1}), g_z(t, s-1, x_{s-2})) \quad (49)$$

$$K_C(s) = S(y, 0)D_s + S(y+1, 0) = f_{K_C(s)}(D_s) \quad (50)$$

$$K_O^Z(s) = \frac{1}{a_s a_{s-1}} \left\{ \begin{array}{c} g_z(t, s-1, x_{s-1}) \left[\frac{S(y,0)D_{s-2}a_{s-1}+(a_{s-1}-1)a_s S(y+1,0)}{K_C(s-2)} \right. \\ \left. + \frac{S(y,0)g_z(t, s-1, x_{s-2})D_{s-3}(a_{s-1}-1)a_s}{K_C(s-3)} \right] \\ \left. + \frac{S(y+1,0)a_{s-1}}{K_C(s-1)} \right\} \\ = f_{K_O^Z(s)}(D_{s-1}, D_{s-2}, D_{s-3}, g_z(t, s-1, x_{s-1}), g_z(t, s-1, x_{s-2})) \quad (51)$$

$$K_V^Z(s) = 1 + g_z(t, s-1, x_{s-1}) \left\{ \begin{array}{c} \frac{S(y,0)D_{s-2}}{K_C(s-2)} \\ + \frac{a_{s-1}-1}{a_{s-1}} \left(\frac{S(y,0)D_{s-3}}{K_C(s-3)} g_z(t, s-1, x_{s-2}) + \frac{S(y+1,0)}{K_C(s-2)} \right) \end{array} \right\} \\ = f_{K_V^Z(s)}(D_{s-2}, D_{s-3}, g_z(t, s-1, x_{s-1}), g_z(t, s-1, x_{s-2})) \quad (52)$$

$$f^Z(t, s) = \frac{F(s)^-}{C(s)} = \sum_{j=t}^{s-1} (1 - g_z(j)) K_O^Z(j) \frac{e^{\sum_{i=j+1}^s r_i}}{\prod_{g=1}^{s-j} I(j+g)} \quad (53)$$

The following is the expression for the notional factor:

$$I(t) = S_t D_{t-1} \frac{S(y, 0)D_t + S(y+1, 0)}{S(y, 0)D_{t-1} + S(y+1, 0)} \quad (54)$$

$$I(t+1) = S_{t+1} D_t e^{\delta} \frac{S(y, 0)D_{t+1} + S(y+1, 0)}{S(y, 0)D_t + S(y+1, 0)} \quad (55)$$

for the first two periods of time $t, t+1$, and for $j \geq 2$ and $j \leq -1$,

$$I(t+j) = S_{t+j} D_{t+j-1} \frac{S(y, 0)D_{t+j} + S(y+1, 0)}{S(y, 0)D_{t+j-1} + S(y+1, 0)} = f_{I(t+j)}(S_{t+j}, D_{t+j}, D_{t+j-1}) \quad (56)$$

The δ affecting the demographic process at time t represents a one-time exogenous shock. Furthermore, we assume that the individual increments have zero expected value, i.e., $E[w_x(j) - w_x(k)] = 0$ for $x = \gamma, R$ and $j \neq k$, and, finally, that the increments of both processes are independent, i.e., $E[(w_R(j) - w_R(k))(w_\gamma(j) - w_\gamma(k))] = 0$ for $j \neq k$. These assumptions leads to $Cov(D_j, D_k) = 0$ and $Cov(D_j, S_k) = 0$ for $j \neq k$ and to a non-zero covariance when the processes interact during the same period

$$Cov(S_s, D_s) = e^{R+\gamma + \frac{\sigma_R^2 + \sigma_\gamma^2}{2}} (e^{\rho\sigma_R\sigma_\gamma} - 1).$$

The automatic balance mechanism at time $s = t$ depends on the random variable vector $X_t = (D_{t-3}, D_{t-2}, D_{t-1})$ for $z=LR$ and $X_t = (D_{t-3}, D_{t-2}, D_{t-1}, D_t)$ for $z=SR$, and on the random variable vector $X_s = (D_{t-3}, \dots, D_t, S_t, \dots, S_s)$ for $s \geq t + 1$ for both $z=LR, SR$ with $n = 2(s - t) + 4$ length of the vector.

The following represents the joint distribution of a random vector X_s :

$$f_{X_s}(x) = \prod_{j=t-3}^t f_{D_i}(d_i) \prod_{j=t+1}^s f_{S_j, D_j}(s_j, d_j) \quad (57)$$

for $s \geq t + 1$ where the distribution of the product of the demographic process D_s and salary process $S_s \forall s$ is denoted by $S_s D_s \sim \log N(R + \gamma - \frac{\sigma_R^2 + \sigma_Y^2}{2}, \sigma_{R,Y}^2)$ with $\sigma_{R,Y}^2 = \sigma_R^2 + \sigma_Y^2 + 2\rho\sigma_R\sigma_Y$. The following represents the joint density function of (S_s, D_s) is:

$$f_{S_s, D_s}(x, y) = \frac{1}{xy\sqrt{|\Sigma|}} e^{-\frac{1}{2|\Sigma|} \left((\log z - \mu)' \Sigma^{-1} (\log z - \mu) \right)} \text{ for } xy > 0 \quad (58)$$

with:

$$\log z = \begin{pmatrix} \log x \\ \log y \end{pmatrix}, \mu = \begin{pmatrix} R - \frac{\sigma_R^2}{2} \\ \gamma - \frac{\sigma_Y^2}{2} \end{pmatrix} \quad (59)$$

$$\Sigma = \begin{pmatrix} \sigma_Y & \rho\sigma_Y\sigma_R \\ \rho\sigma_Y\sigma_R & \sigma_R \end{pmatrix} \quad (60)$$

$$|\Sigma| = \text{determinant of variance-covariance matrix } \Sigma \quad (61)$$

Finally, the k^{th} raw moment of the the automatic balance mechanism for $z=LR, SR$ is given given by the following:

$$E[B_z(s)^k] = E[g_z(t, s, x_s)^k] = \int_0^\infty \dots \int_0^\infty g_z(t, s, x_1, \dots, x_n)^k f_X(x_1, \dots, x_n) dx_1 \dots dx_n \quad (62)$$

The transformation $g_z(t, s, x_s)$ when the automatic balance mechanism sought is symmetric is given by the following:

$$g_{LR}(t, s, x_s) = \frac{1 + f^{LR}(t, s)}{K_O^{LR}(s)} \quad (63)$$

$$g_{SR}(t, s, x_s) = \frac{TD^{SR}(s) + f^{SR}(t, s)}{K_V^{LR}(s)} \quad (64)$$

However, if the automatic balance mechanism is asymmetric, the transformation is given by the following:

$$g_{LR}(t, s, x_s) = \text{Min} \left[\frac{1 + f^{LR}(t, s)}{K_O^{LR}(s)}, 1 \right] \quad (65)$$

$$g_{SR}(t, s, x_s) = \text{Min} \left[\frac{TD^{SR}(s) + f^{SR}(t, s)}{K_V^{LR}(s)}, 1 \right] \quad (66)$$

Note that the function $g_{LR}(t, s, x_s)$ reduces to $\frac{1}{K_O^{LR}(s)}$ if the fund equals 0 when the automatic balance mechanism is first applied and if the automatic balance mechanism is symmetric.

As denoted in the section ‘A dynamic four overlapping generations model’, the formulas were developed for cases in which longitudinal equilibrium, also known as actuarial fairness, is a constraint. This constraint implies that different pensioner cohort would face different pension indexation according to the realized mortality and economic outcomes. Nevertheless, in practice, this solution is not politically viable. Therefore, we consider a second case without the actuarial fairness constraint by using current mortality data rather than projected mortality values. Throughout the paper, this is called Case 2. In this case, the expressions representing the outcome $O(t)$ (18), turnover duration $TD(t)$ (25) and liabilities $V(t)$ (21) slightly change as follows:

$$K_O^Z(t) = \frac{K_N^Z(t)}{a_t} + K_N^Z(t-1) \frac{a_t - 1}{a_{t-1}} \quad (67)$$

$$K_V^Z(t) = 1 + B_z(t-1) \left\{ \frac{S(y, 0)e^{R_{t-2}}}{K_C(t-2)} + K_N(t-1) \frac{a_t - 1}{a_{t-1}} \right\} \quad (68)$$

$$TD^Z(t) = 2 + B_z(t-1) \frac{K_N^X(t-1)}{K_O^X(t)} \frac{a_t - 1}{a_{t-1}} - \frac{S(y+1, 0)}{K_C(t)} \quad (69)$$

where $K_N^Z(t)$ is the same as in (22). Note that the main difference between the Case 1 and Case 2 is that the ratio $\frac{a_{t-1}-1}{a_{t-1}}$ becomes $\frac{a_t-1}{a_{t-1}}$. The calculation of the variances and expected values participants parallels Case 1.