

Arbitrage and equilibrium with exchangeable risks ¹

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Abstract

In an economy with a non - atomic measure space of assets and exchangeable risks, the Arbitrage Pricing Theory (APT) holds exactly; and factors are structurally specified, which allows for an economic interpretation.

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1 Introduction

The Arbitrage Pricing Theory (APT) of Ross (1976) introduced the idea that few factors govern the prices of assets: if no arbitrage opportunities exist, then the risk premium on any asset is determined by the factor loadings of the asset; the remaining loadings of the idiosyncratic residuals bear (almost) no risk premium, since idiosyncratic risk can be diversified in a large portfolio.

Uncertainty is described by a probability space¹; when not necessary, the subscript that indicates the measure is omitted; the space of square - integrable functions is a self - dual hilbert space.

There are countably many assets, with payoffs $x_n : n = 1, \dots$, square - integrable random variables². The payoff of every asset takes the form

$$x_n = \sum_{k=1}^K b_{n,k} f_k + \varepsilon_n,$$

where $f_k : k = 1, \dots, K$ are factors, finitely many; the residuals, $\varepsilon_n : n = 1, \dots$, have zero mean: $E(\varepsilon_n) = 0$, unit variance: $\text{Var}(\varepsilon_n) = 1$, and are mutually uncorrelated: $E(\varepsilon_n \varepsilon_{\hat{n}}) = 0$.

The prices of assets do not allow for arbitrage: the price functional, p , on the domain of square - integrable random variables, is linear, positive and continuous.

By the riesz representation theorem³, there exists a random variable, m^* , such that

$$p(x_n) = E(x_n m^*).$$

The random variable m^* is the payoff of a benchmark portfolio, with the property that assets are priced by their inner product with m^* . The existence of a benchmark portfolio is the abstract version of the Capital Asset Pricing Model (CAPM). The payoff of the benchmark portfolio can be chosen in the closure of the span of the payoffs of assets. If the preferences of individuals depend only on the mean and variance of payoffs, the benchmark portfolio is collinear with aggregate demand⁴.

By the bessel inequality⁵,

$$p(m^*) = E(m^{*2}) \geq \sum_{n=1}^{\infty} |E(m^* \varepsilon_n)| = \sum_{n=1}^{\infty} |p(\varepsilon_n)|.$$

¹Expectations with respect to a measure, ν , are “ E_ν ,” variances “ Var_ν ” and covariances “ Cov_ν .”

²It is pedantic to distinguish between portfolios and assets and their payoffs, and, similarly, for factors and residuals

³Royden (1968; ch. 6, thm. 8), Dudley (1989; stm. 6.4.1).

⁴Lintner (1965), Sharp (1964), Traynor (1961).

⁵Dudley (1989; stm. 5.4.3).

It follows that, for any $\delta > 0$, the number of residuals whose prices exceed δ in absolute value is finite. By the linearity of the price functional ⁶,

$$p(x_n) = \sum_{k=1}^K b_{n,k} p(f_k) + p(\varepsilon_n),$$

and, hence, the number of assets whose prices differ from the prices determined by the factor loadings by more than δ is finite. Nevertheless, though finite, the number of assets whose price differ substantially from the price determined by the factor loadings can be large. In this sense, the framework of Ross (1976) delivers the desired result only in part.

An advantage of the framework is that the factors are a priori specified; which allows for structural models. A weakness of the framework is that the factor structure is not a characteristic of the subspace spanned by the payoffs of assets: it is possible that there exist payoffs of assets $y_k : k = 1, \dots$, whose span coincides with the span of $x_n : n = 1, \dots$, that do not admit the same factor structure or a factor structure whatsoever ⁷.

In recent papers, Al - Najjar (1994, 1995, 1999, a) address these issues. He has shown that a non - atomic measure space of traded assets yields prices that coincide with the prices determined by the factor loadings for almost all assets: the APT holds exactly — almost all the residuals have zero price ⁸; and the factor structure, which is of countable dimension, is determined by the span of the payoffs of assets and, thus, is intrinsic ⁹.

The APT holds exactly also in the framework of Werner (1997) with portfolios that are finitely additive spreads over a countable set of assets.

An alternative formulation by Khan and Sun (1997, a, b) and Sun (1997) with assets indexed by a non - standard model of the natural numbers, allows for a more direct transfer of arguments between large finite economies and the limit economy, where the APT holds exactly.

Importantly, whenever the APT holds exactly, the benchmark portfolio of the CAPM is in the closure of the span of the factors; which draws a link between the two arguments.

In the exact constructions of the APT above, the space of factors, even if intrinsic, remains a derived construct; while the basic motivation that underlies the APT is the possibility of accounting for the prices of assets by few and economically transparent factors.

⁶This assumes, implicitly, that the factors, and hence the residuals, are traded; if factors are not traded, it suffices to replace $p(f_k)$ by $E(f_k m^*)$, and $p(\varepsilon_n)$ by $E(\varepsilon_n m^*)$.

⁷Bray (1994, a, b).

⁸Khan and Sun (1997, c) provide a counterexample that casts doubts on the generality of the approach; Al Najjar (1999, b) responds.

⁹An extension of factor analysis to spaces of countably infinite dimension specifies a basis for the space of factors: the first factor accounts for a maximum of the variance of the payoffs of assets, and each subsequent factor accounts for a maximum of the variance of payoffs projected on the orthogonal complement of the factors already specified.

Here, a particular case of an economy with exchangeable risks allows factors to be specified a priori and be interpreted as aggregate risks. A collection of random variables is exchangeable if, for every natural number, k , the joint distribution of any k of the variables is the same. By a theorem of de Finetti ¹⁰, an exchangeable distribution is the result of choosing a vector - valued parameter according to a given distribution, and then choosing an i.i.d. process according to the realized parameter.

With a continuum of assets with exchangeable payoffs, portfolios are (possibly non - atomic) spreads; non - arbitrage refers to the linearity and positivity of the price function, and not to asymptotic properties or continuity ¹¹; and the APT holds exactly.

The claim that the a priori specification of factors allows for interpretation, in particular as aggregate risks, can be made precise only in the framework of a fully specified model of general equilibrium. The models of Arrow and Lind (1970) and Malinvaud (1972, 1973) of large economies with ex - ante identical individuals, and of the variant in Cass, Chichilnisky and Wu (1995) that allows for heterogeneity, can be extended to economies with exchangeable risks and portfolios that may be non - atomic, to yield exact limit results.

2 Arbitrage and equilibrium in a simple economy

The economy is finite: the set of states of the world is S , a finite set.

There is one commodity; a bundle of commodities across states of the world is ¹² $x = (\dots, x_s, \dots)'$.

An asset or a portfolio is described by its payoffs across states of the world. A basis for the payoffs of assets is formed by the payoffs of primary assets, $j \in J$, a finite set; the matrix of payoffs of primary assets is $A = (\dots, x_j, \dots)$, and the space of portfolios is ¹³ $M = [A]$; holdings of primary assets are $\theta = (\dots, \theta_j, \dots)'$.

Prices of assets are p , a function from the set of portfolios to the real numbers; prices of primary assets are $q = (\dots, q_j, \dots)$; implicit prices of revenue across states of the world are $\pi = (\dots, \pi_s, \dots)$, and can be interpreted as a measure on S .

The set of states of the world has a product structure: $S = F \times N$, and $s = (f, n)$, where F is a non - empty, finite set of states of aggregate uncertainty, and N is a non - empty, finite set of states of idiosyncratic uncertainty; that

¹⁰Hewitt and Savage (0000; thm 7.2).

¹¹Gilles and Leroy (1991) point out that continuity of the price functional in the L^2 norm need not be appropriate in the definition of non - arbitrage if individuals care about higher moments of the distribution of returns and their utility functions are not continuous in the L^2 norm.

¹²"'" denotes the transpose

¹³"["] denotes the column span of a matrix or of a collection of vectors.

the set of states of idiosyncratic uncertainty does not vary with the state of aggregate uncertainty is without loss of generality.

A portfolio writes as $m = (\dots, m_f, \dots)$, where $m_f = (\dots, m_{f,n}, \dots)$; similarly, implicit prices of revenue, $\pi = (\dots, \pi_f, \dots)$, where $\pi_f = (\dots, \pi_{f,n}, \dots)$;

The payoffs of an asset are invariant with respect to idiosyncratic uncertainty if $m_{f,n} = \bar{m}_f$ and, similarly, for a bundle of commodities.

There is a probability measure, μ , over the set of states of the world that decomposes into, ν , a probability measure over the set of states of aggregate uncertainty, and probability measures on the set of states of idiosyncratic uncertainty, μ_f , conditional on aggregate uncertainty: $\mu_{f,n} = \mu_{n|f}\nu_f$.

Implicit prices of revenue are risk neutral with respect to idiosyncratic uncertainty if $\pi_{f,n} = \mu_{n|f}\pi_f$, where π_f , are implicit prices of revenue across states of aggregate uncertainty.

A portfolio bears only idiosyncratic uncertainty if its expectation conditional on aggregate uncertainty vanishes: $E_{\mu_f} m_f = 0$, for all $f \in F$.

The space of portfolios has an orthogonal decomposition:

$$M = M_F + M_I, \quad \text{and} \quad M_F \perp M_I = 0,$$

where assets $m \in M_F$ are factors, with payoffs invariant with respect to idiosyncratic uncertainty: $m_{f,n} = m_f$, and assets $m \in M_I$ bear only idiosyncratic uncertainty: $E_{\mu_f} m = 0$; a portfolio decomposes as

$$m = m_F + m_I.$$

There is no presumption that ¹⁴ $\dim M_F = \#F$, in which case the market for factors would be complete.

The preferences of an individual, reflect the factor structure if the consumption set is $X = \times_{f \in F} X_f$, where $X_f = \times_{n \in N} C_f$, and the preference relation is represented by a utility function that is monotonically increasing, strictly quasi-concave and separable across factors: $u^i = \sum_{f \in F} u_f^i$, where, $u_f^i = E_{\mu_f} v_f^i$, is a von neumann - morgenstern representation conditional on aggregate uncertainty.

If the separable utility function is smooth at a consumption plan that is invariant with respect to idiosyncratic uncertainty, then the gradient, Du , is risk - neutral with respect to idiosyncratic uncertainty: $Du_{f,n} = \pi_f \mu_{f|n}$; the argument extends to implicit prices of revenue that support the consumption plan if the utility representation is not smooth.

The prices of assets p , do not allow for arbitrage if they are linear:

$$p(m) = q\theta, \quad \text{where} \quad m = A\theta,$$

and value positively limited liability portfolios:

$$m > 0 \quad \Rightarrow \quad p(m) > 0.$$

¹⁴“#” denotes the cardinality of a set.

By a theorem of Ross (1978), the prices of assets p do not allow for arbitrage if and only if

$$q = \pi A, \quad \pi \gg 0.$$

By the abstract version of the CAPM, there exists a benchmark portfolio, with payoff m^* , such that

$$p(m) = E_\mu(mm^*).$$

Proposition 1 *If the prices of assets, p , do not allow for arbitrage and the implicit prices of revenue: $q = \pi A$ are risk - neutral with respect to idiosyncratic uncertainty:*

$$\pi_{f,n} = \mu_{n|f}\pi_f,$$

where $\pi_{\mathcal{F}}$, are implicit prices of revenue across states of aggregate uncertainty, then the benchmark portfolio is the payoff of a portfolio of factors:

$$m^* \in M_F,$$

and

$$m = m_F + m_I \quad \rightarrow \quad p(m) = p(m_F).$$

Proof The benchmark portfolio is

$$m^* = A(E_\mu(A'A))(E_\pi A').$$

Without loss of generality, $A = (A_F; A_I)$, where $[A_F] = M_F$ and $[A_I] = M_I$. Since $E_\mu(A'_F A_I) = 0$, the payoff of the benchmark portfolio rewrites as

$$m^* = A_F(E_\mu(A'_F A_F))(E_\pi A'_F).$$

□

This is the pricing equation of APT.

In an economy with aggregate endowment that is invariant with respect to idiosyncratic uncertainty, the consumption bundles of pareto optimal allocations inherit this invariance. With an asset market that is sufficiently rich if not complete, competitive equilibrium allocations are pareto optimal. Implicit prices of revenue that support pareto optimal competitive equilibrium allocations are invariant with respect to idiosyncratic uncertainty and, by the proposition above, the prices of assets satisfy the APT pricing equation. This is the argument in Arrow and Lind (1970), Cass, Chichilnisky and Wu (1996), Cass and Shell (1983) and Malinvaud (1972, 1973).

In a finite economy, APT pricing requires equilibrium considerations that go beyond no - arbitrage. This is restrictive, especially in comparison with the argument in a large economy; but, importantly, the pricing relation applies to all assets.

3 Arbitrage

The economy has a product structure.

The state space is $S = \prod_{r \in [0,1]} T_r$, where each T_r is a copy of T , a finite set of possible types. Each T_r is equipped with the discrete topology, and S is equipped with the borel σ - field of the product topology. The set of regular borel probability measures on S is $\Delta(S)$.

The T - valued random variable on S that gives the value of the r - coordinate is x_r , and

$$x_{r,t} = \begin{cases} 1, & \text{if } x_r = t, \\ 0, & \text{otherwise.} \end{cases}$$

The simplex of probability measures on T is $\Delta(T)$. For every $q \in \Delta(T)$, $q^{[0,1]}$ is the independent product probability measure on S .

There is a probability measure μ on S , which is exchangeable: for every finite subset of indices, $k_1, \dots, k_n \in [0, 1]$, any permutation of them, m_1, \dots, m_n , and any $t_1, \dots, t_n \in T$,

$$\mu[x_{k_1} = t_1, \dots, x_{k_n} = t_n] = \mu[x_{m_1} = t_1, \dots, x_{m_n} = t_n].$$

According to the theorem of de Finetti, the measure, $\mu \in \Delta(S)$ is exchangeable if and only if there exists a probability measure ν on $\Delta(T)$, such that, for every measurable $E \subseteq S$,

$$\mu(E) = \int_{\Delta(T)} \left(\int_S q^{[0,1]}(E) \right) d\nu(q).$$

Thus, an exchangeable distribution μ is the result of drawing — according to ν — a probability measure q on T , and then drawing independently from T , according to q , infinitely many times.

For $q \in \Delta(T)$,

$$[q] = \bigcap_{k_1, k_2, \dots \in [0,1]} \left\{ s \in S : \lim_{n \rightarrow \infty} \frac{\#\{k_m \in \{k_1, \dots, k_n\} : X_{k_m}(s) = t\}}{n} = q(t), \quad t \in T \right\}.$$

By the strong law of large numbers ¹⁵, $q^{[0,1]}([q]) = 1$; as it is the intersection of closed sets, $[q]$ is closed, it is the support of $q^{[0,1]}$.

For every $t \in T$, the factor F_t is the random variable defined by

$$f_t((s)) = \begin{cases} q(t), & \text{if } s \in [q] \\ 0, & \text{otherwise.} \end{cases}$$

¹⁵Dudley (1989; stm. 0.0.0).

For every $t \in T$, $\int_S F_t(s) d\mu > 0$; otherwise one deletes t from T .

An asset is a real - valued random variable that specifies the payoff in every state $s \in S$. There are two types of assets in the set, A , of primary assets: The first are finitely many factors, f_t , for $t \in T$. The second are linear combinations of the form

$$y_r = \sum_{t \in T} c_{r,t} x_{r,t}, \quad r \in [0, 1],^{16}$$

with coefficients $c_r = (c_{r,t} : t \in T)$ in a fixed, finite set $C = \{c^1, \dots, c^N\}$, where each $I^n = \{r \in [0, 1] : c_r = c^n\}$, is a measurable set of indices. The set A can be identified with $T \cup [0, 1]$. A portfolio is $m \in M$, a bounded, signed measure on the set of primary assets. By the Jordan decomposition theorem¹⁷, every portfolio has a unique decomposition, $m = m^+ - m^-$, where m^+ and m^- are positive, mutually singular measures; m^+ represents purchases and m^- sales of primary assets.

The portfolio $\sum_{t \in T} f_t$ pays off 1 μ - almost surely, and is, thus, an essentially riskless asset.

Remark The main difference between the model of Al - Najjar (1999, a) and the model here is that the former only allows for portfolios that are finite linear combinations of the set of primary assets. With a continuum of assets, it is natural to consider purchases and sales of a continuum of assets, as is suggested in Al - Najjar (1995).

For $r \in [0, 1]$,

$$z_r = y_r - \sum_{t \in T} c_{r,t} f_t = \sum_{t \in T} c_{r,t} (x_{r,t} - f_t).$$

Lemma 1

$$E_\mu(z_r) = 0.$$

Proof By the theorem of de Finetti, it is sufficient that, for every $q \in \Delta(T)$,

$$E_{q^{[0,1]}}(z_r) = 0.$$

Therefore, it is sufficient that, for every $t \in T$,

$$E_{q^{[0,1]}}(x_{r,t} - f_t) = 0.$$

Indeed, $E_{q^{[0,1]}} x_{r,t} = q(t)$, while, by the strong law of large numbers, $E_{q^{[0,1]}} f_t = q(t)$. \square

¹⁶The notations $y(r, s)$ and $y_r(s)$ are interchangeable; $y_r : S \rightarrow \Re$.

¹⁷Dudley (1989; stm. 5.6.1)

Lemma 2

$$\text{Cov}(z_r, z_{\hat{r}}) = 0, \quad r \neq \hat{r}.$$

Proof By proposition 1, it suffices to prove that

$$\mathbb{E}_\mu(z_r z_{\hat{r}}) = \mathbb{E}_\mu(z_r) \mathbb{E}_\mu(z_{\hat{r}}) = 0.$$

By the theorem of de Finetti, it is sufficient to prove that, for every $q \in \Delta(T)$,

$$\mathbb{E}_{q^{[0,1]}}(z_r z_{\hat{r}}) = 0;$$

equivalently, that

$$\mathbb{E}_{q^{[0,1]}}\left(\sum_{t \in T} c_{r,t}(x_{r,t} - f_t)\right) \left(\sum_{t \in T} c_{\hat{r},t}(x_{\hat{r},t} - f_t)\right) = 0.$$

By the strong law of large numbers, $f_t = q(t)$, $q^{[0,1]}$ - almost everywhere. Thus, it remains to show that

$$\mathbb{E}_{q^{[0,1]}}\left(\sum_{t \in T} c_{r,t}(x_{r,t} - q(t))\right) \left(\sum_{t \in T} c_{\hat{r},t}(x_{\hat{r},t} - q(t))\right) = 0.$$

But, according to $q^{[0,1]}$, the random variables $\sum_{t \in T} c_{r,t}(x_{r,t} - q(t))$ and $\sum_{t \in T} c_{\hat{r},t}(x_{\hat{r},t} - q(t))$ are independent, with zero expectation. Hence, the expectation of their product, with respect to $q^{[0,1]}$, is equal to the product of their expectations, which vanishes. \square

At this point one would like to define the state - wise payoffs of a portfolio, m , as the integral, according to m , of the state - wise payoffs of the primary assets in A . However, for most states, $s \in S$, the function $r \rightarrow Y_r(s)$ is not measurable, and the lebesgue integral is not well defined. Therefore, one resorts to the notions of weak measurability and Pettis integration ¹⁸.

A function $y : [0, 1] \times S \rightarrow \mathfrak{R}$ is weakly measurable with respect to $q^{[0,1]}$ if, given any square - integrable function $h : S \rightarrow \mathfrak{R}$, the function

$$r \rightarrow \int_S h(s) y(r, s) dq^{[0,1]}(s)$$

is measurable.

A weakly measurable function $y : [0, 1] \times S \rightarrow \mathfrak{R}$ is Pettis - integrable with respect to $q^{[0,1]}$ and m if there exists a square - integrable function $g : S \rightarrow \mathfrak{R}$, such that, for any square -integrable function $h : S \rightarrow \mathfrak{R}$,

$$\int_0^1 \left[\int_S h(s) y(r, s) dq^{[0,1]}(s) \right] dm(r) = \int_S h(s) g(s) dq^{[0,1]}(s).$$

¹⁸Pettis (1938)

The function g is the pettis integral of y with respect to $q^{[0,1]}$ and m , and one writes

$$g(s) = \int_0^1 y(r, s) d_q^* m(r).$$

Since the space of square - integrable functions is self - dual, every bounded, weakly measurable y is also pettis integrable: the map f , defined by

$$f(h) = \int_0^1 \left[\int_S h(s) y(r, s) dq^{[0,1]}(s) \right] dm(r)$$

is bounded and linear, so by the riesz representation theorem there exists a square - integrable function, g , such that

$$f(h) = \int_S h(s) g(s) dq^{[0,1]}(s),$$

for every square -integrable function.

The functions y are weakly measurable and therefore also pettis - integrable. Indeed, in every equivalence class of $q^{[0,1]}$ - almost everywhere equal functions, is a baire measurable function, h , which depends on at most countably many coordinates. Thus, for λ - almost every $r \in [0, 1]$ — those r on which h does not depend,

$$\begin{aligned} \int_S h(s) y_r(s) dq^{[0,1]}(s) &= \\ \int_S h(s) dq^{[0,1]}(s) \int_S y_r(s) dq^{[0,1]}(s) &= \\ \int_S h(s) dq^{[0,1]}(s) \int_S \sum_{t \in T} c_{r,t} x_{r,t}(s) dq^{[0,1]}(s) &= \\ \sum_{t \in T} c_{r,t} q(t) \int_S h(s) dq^{[0,1]}(s). \end{aligned}$$

Since the map $r \rightarrow c_r$ is measurable, so is $r \rightarrow \int_S h(s) y_r(s) dq^{[0,1]}(s)$, and, hence, the function y is weakly measurable, as required. Furthermore, if m is non - atomic,

$$\int_0^1 y(r, s) d_q^* m(r) = \sum_{t \in T} q(t) \left(\int_0^1 c_{r,t} dm(r) \right).$$

This is because

$$\begin{aligned} \int_0^1 \left[\int_S h(s) y_r(s) dq^{[0,1]}(s) \right] dm(r) &= \\ \int_0^1 \left(\sum_{t \in T} c_{r,t} q(t) \int_S h(s) dq^{[0,1]}(s) \right) dm(r) &= \\ \left[\int_S h(s) dq^{[0,1]}(s) \right] \left[\sum_{t \in T} q(t) \left(\int_0^1 c_{r,t} dm(r) \right) \right] &= \\ \int_S h(s) \left[\sum_{t \in T} q(t) \left(\int_0^1 c_{r,t} dm(r) \right) \right] dq^{[0,1]}(s) \end{aligned}$$

The de finetti representation of μ defines the state - wise payoffs of a non -atomic portfolio, m , by

$$\bar{m}(s) = \begin{cases} \int_0^1 y(r, s) d_q^* m(r), & \text{if } s \in [q] \\ 0, & \text{otherwise.} \end{cases}$$

If m is atomic, $\bar{m}(s)$ is just a weighted sum of finitely or countably many of the primary assets payoffs in s . Since every portfolio the sum of an atomic portfolio and a non - atomic portfolio, state - wise payoffs are well defined

A price functional is a measurable function, p , from the set of portfolios, M , to the real numbers. The return of a portfolio, m , with positive price is the random variable defined by

$$r_m(s) = \frac{\bar{m}(s)}{p(m)} - 1.$$

The expected return of m with respect to μ is $\bar{r}(m)$. The return of the risk - free portfolio, $\sum_{t \in T} f_t$, which pays 1 μ - almost surely, is r_f . The risk premium of the portfolio m is the difference $\bar{r}(m) - r_f$.

The price functional, p , admits no arbitrage if it is linear:

$$p(m) = \int_A p dm,$$

and it values positively limited liability portfolios:

$$\bar{m}(s) \geq 0, \text{ and } \mu(\bar{m}(s) > 0) > 0 \Rightarrow p(m) > 0.$$

Proposition 2 *If the price functional admits no arbitrage, then*

$$p(y_r) = \sum_{t \in T} c_{r,t} p(f_t), \quad \lambda - a.e \ r \in [0, 1],$$

and

$$\bar{r}(y_r) - r_f = \sum_{t \in T} \frac{c_{r,t} p(f_t)}{\sum_{i \in T} c_{r,i} p(f_i)} (\bar{r}(f_t) - r_f), \quad \lambda - a.e \ r \in [0, 1].$$

Proof If there is a measurable set of indices, $E \subseteq [0, 1]$, with positive lebesgue measure, such that $p(z_r) < 0$, for $r \in E$, (the argument for the case where $p(z_r) > 0$, for $r \in E$, is analogous) then, for any non - atomic probability measure, m , on E , according to the definition of no - arbitrage,

$$p(m) = \int_E p(y_r) dm(r) <$$

$$\int_E (\sum_{t \in T} c_{r,t} p(f_t)) dm(r) = \sum_{t \in T} p(f_t) \left(\int_0^1 c_{r,t} dm(r) \right).$$

It follows that

$$\bar{m}(s) = \begin{cases} \sum_{t \in T} q(t) \left(\int_0^1 c_{r,t} dm(r) \right), & \text{if } s \in [q] \\ 0, & \text{otherwise} \end{cases}$$

Therefore, going long on m and going short $\left(\int_0^1 c_{r,t} dm(r) \right)$ on f_t , for each $t \in T$, leaves a positive revenue to invest in the risk - free asset, thus generating a zero - cost, limited liability portfolio, which violates the condition of no - arbitrage. The pricing equation follows.

Since the f_t are limited liability assets, by no - arbitrage, $p(f_t) > 0$. By lemma 1 and corollary 1,

$$\begin{aligned} \bar{r}(y_r) &= \\ &= \frac{E_\mu \sum_{t \in T} c_{r,t} f_t + E_\mu z_r^c}{\sum_{t \in T} c_{r,t} p(f_t)} - 1 = E_\mu \left(\frac{\sum_{t \in T} c_{r,t} f_t}{\sum_{t \in T} c_{r,t} p(f_t)} - 1 \right) = \\ &= E_\mu \left(\sum_{t \in T} \frac{c_{r,t} P(f_t)}{\sum_{i \in T} c_{r,i} p(f_i)} \left(\frac{f_t}{p(f_t)} - 1 \right) \right) = E_\mu \sum_{t \in T} \frac{c_{r,t} p(f_t)}{\sum_{i \in T} c_{r,i} p(f_i)} \bar{r}(f_t) \\ &= \lambda \text{ - a.e. } r \in [0, 1]. \end{aligned}$$

The weights $c_{r,t} p(f_t) / (\sum_{i \in T} c_{r,i} p(f_i))$ sum to 1. Subtracting r_f from each side yields the excess return. \square

The price of almost every primary asset is determined by the factor loadings and the excess return satisfies the APT expression of the CAPM equation.

In a large economy with exchangeable risks, APT pricing does not require equilibrium considerations that go beyond no - arbitrage. This is substantially more general than the argument in a finite economy, but, of course, the pricing relation applies to almost all assets.

4 Equilibrium

Individuals are indexed by r in the unit interval with the borel σ -field and λ , the lebesgue measure.

States of the world are $s \in S = \prod_{r \in [0,1]} T_r$, where $T_r = T$ is the finite set of possible types of an individual.

A state of the world realizes according to the exchangeable probability measure μ .

There are L commodities and a bundle of commodities is an element of \mathfrak{R}^L .

An individual is described by a continuous, monotonically increasing and strictly concave cardinal utility index, u^r , defined over non - negative bundles

of commodities, and by a random endowment, e^r , defined by

$$e^r(s) = \sum_{t \in T} x_{r,t}(s) e_t;$$

the type of an individual thus determines his endowment, $e_t \geq 0$.

A consumption plan for an individual is a non - negative, vector - valued random variable, $c^r : S \rightarrow \mathfrak{R}_+^L$; it yields utility $\int_S u^r(c^r) d\mu$.

An allocation of commodities is a weakly measurable function $c : [0, 1] \times S \rightarrow \mathfrak{R}_+^L$. Since, for most values of s , the function $r \rightarrow e^r$ is not measurable, feasibility is defined by means of the Pettis integral. A feasible allocation is such that, $q^{[0,1]}$ - almost everywhere, for $q \in \Delta(T)$,

$$\int_{[0,1]} x(r, s) d_q^* \lambda(r) = \int_{[0,1]} e^r(s) d_q^* \lambda(r).$$

The allocation of endowments $e : [0, 1] \times S \rightarrow \mathfrak{R}_+^L$ is, indeed, weakly measurable.

An allocation, \hat{c} , is Pareto superior to another allocation, c , if

$$\int_S u^r(\hat{c}(r, s)) d\mu(s) > \int_S u^r(c(r, s)) d\mu(s), \quad \lambda - \text{a.e. } r \in [0, 1].$$

A feasible allocation, c , is Pareto optimal if there does not exist any other feasible allocation that is Pareto superior.

An allocation, c , is essentially constant on a measurable set $E \subset S$ if, for λ - a.e. $r \in [0, 1]$, the function $s \rightarrow c(r, s)$ is constant, for μ - a.e. $s \in E$.

Lemma 3 *If an allocation is Pareto optimal, then it is essentially constant on $[q]$, for every $q \in \Delta(T)$.*

Proof Given an allocation, c , the allocation \hat{c} , defined by

$$\hat{c}(r, s) = \int_{[q]} c(r, s) d\mu(s),$$

on $[q]$, for all $q \in \Delta(T)$, and equal to c , otherwise, is weakly measurable, and its Pettis integral coincides with the Pettis integral of c on $[q]$, for every $q \in \Delta(T)$; it is, thus, feasible. If the allocation c is not essentially constant, there exists a non - null set of individuals whose expected utility will be higher at \hat{c} than at c , due to the strict concavity of cardinal utility indices; but this is incompatible with Pareto optimality for c . \square

This is the analogue of the result of Arrow and Lind (1970) and Malinvaud (1972, 1973) for the case of a large finite economy with idiosyncratic risks. It is also a no - sunspot result, in the sense of Cass and Shell (1983). In an exchangeable economy, only aggregate states, indexed by q , are relevant for the definition

of the aggregate endowment. All additional uncertainty, though relevant for the determination of the individuals endowments, is, in this sense, extrinsic. With strict concavity of the utility functions, and the insurance possibilities open to individuals, pareto optimal allocations are determined by the realization of aggregate uncertainty and only that.

Malinvaud (1972, 1973) also shows that in an economy with ex - ante identical individuals subject to idiosyncratic shocks, insurance can substitute for a complete set of elementary securities in the sense of Arrow (1953).

Here, if the utility functions of individuals coincide: $u^r = u$, at a pareto optimum, every individual consumes the average endowment $\sum_{t \in T} e_t q(t)$, conditional on q , the realization of aggregate risk.

The exposition is simplified by considering economies with one commodity: $L = 1$.

A pareto optimal allocation can be attained as a competitive equilibrium allocation with all individuals trading in a mutual fund of primary assets and each individual trading in an insurance contract against the idiosyncratic risk of his endowment; the cardinality of the set of primary assets is finite.

The set of primary assets consists of factors, f_t , for $t \in T$. An insurance contract for an individual is an asset with payoff e^r , the endowment of the individual.

At prices p , an individual faces the budget constraints

$$p(m) = 0,$$

$$c^r(s) = e^r(s) + \bar{m}(s), \quad \mu - \text{a.e. } s \in S;$$

a portfolio, m , that satisfies the budget constraints generates the consumption plan c^r , for the individual.

Lemma 4 *If the price functional admits no arbitrage, for λ - a.e. $r \in [0, 1]$, there exists a zero cost portfolio, \hat{m}^r , that generates the pareto optimal consumption plan, c^r .*

Proof The portfolio, m^r , that puts weight - 1 on e^r and weight e_t on the factor f_t , for every $t \in T$, satisfies $p(\hat{m}^r) = 0$ and generates the consumption plan c^r , for λ - a.e. $r \in [0, 1]$. \square

If individuals are not ex - ante identical, one cannot in general achieve the decentralization of a pareto optimum using only the primary assets considered above. An interesting case is the one considered by Cass, Chichilnisky and Wu (1996), in which there are a finite number, N , of different classes of ex - ante identical individuals.

There is a partition, $\{I_n : n = 1, \dots, N\}$ of the set of individuals, $[0, 1]$, into measurable subsets of positive measure.

For every n , all the individuals, $r \in I_n$, have the same utility index, u^n , and their endowments are defined by $e^r(s) = \sum_t x_{r,t}(s) e_t^n$.

A competitive equilibrium allocation is \hat{c} , and prices are \hat{p} ; they are represented by $\hat{\pi}$, such that

$$\hat{p}(c^r) = \int_S \hat{\pi}(s) c^r(s) d\mu(s).$$

Since competitive equilibrium allocations are pareto optimal, for each $q \in \Delta(T)$, they are essentially constant on $[q]$, for λ - a.e $r \in [0, 1]$; for these r , $\hat{c}^r(q)$ denotes $\hat{c}^r(s)$, for $s \in [q]$.

Lemma 5 *The function $q \rightarrow \hat{c}^r(q)$ is measurable on $\Delta(T)$.*

Proof For $k_1, k_2, \dots \in [0, 1]$, any sequence of numbers, $S_\infty = \prod_{j=1}^\infty T_{k_j}$ is a complete, separable metric space.

For the rest of the argument, each $[q]$ is identified with its projection on S_∞ ; similarly, since \hat{c}^r depends only on S_∞ , up to μ - null subsets, \hat{x}^r is treated as defined on S_∞ .

The correspondence from $\Delta(T)$ to S_∞ defined by $q \rightarrow [q]$ admits a measurable selection. The composition of this measurable selection with the measurable map $\hat{c}^r : S_\infty \rightarrow \mathfrak{R}$ yields the function defined by $q \rightarrow \hat{c}^r(q)$, which is therefore measurable as required.

The claim follows from the measurable selection theorem in Hildendrand (1974; thm. 1). By this theorem, the existence of a measurable selection is guaranteed if the graph of the correspondence $q \rightarrow [q]$ is measurable. And indeed, if $(P_i : i = 1, \dots)$ is a sequence of refining partitions of $\Delta(T)$ that separates the points of $\Delta(T)$: P_{i+1} refines P_i for every i , and, for every pair of points in $\Delta(T)$, there is a large enough i , such that P_i separates them, then the graph of $q \rightarrow [q]$ is exactly

$$\left\{ (q, [q]) \in \bigcap_{i=1}^\infty \bigcup_{A \in P_i} \left(A, \bigcap_{k_1, k_2, \dots \in [0, 1]} \left\{ s \in S : \lim_{n \rightarrow \infty} \left(\frac{\#\{k_m \in \{k_1, \dots, k_n\} : t_{k_m}(s) = t\}}{n} \right)_{t \in T} \in A \right\} \right) \right\},$$

which is measurable, as required. \square

To decentralize the allocation \hat{c} using the asset market, individuals trade elementary securities, as in Arrow (1953), on aggregate risks.

More precisely, for each $q \in \Delta(T)$, the characteristic function of the event $[q]$ is χ_q . The set of primary assets A consist of the elementary securities χ_q , for all $q \in \Delta(T)$, and, for all $r \in [0, 1]$, of the endowments e^r . As before, p is a pricing functional on the space of portfolios of primary assets.

Proposition 3 *If $(\hat{c}, \hat{\pi})$ is a competitive equilibrium, there exists a no - arbitrage pricing functional \hat{p} , such that, for λ - a.e. $r \in [0, 1]$,*

1. *there exists a zero cost portfolio, \hat{m}^r , that generates \hat{c}^r , and*

2. the portfolio \hat{m}^r is an optimal choice at prices \hat{p} .

Proof The budget constraint of individual r at (\hat{c}, \hat{p}) is

$$\int_S \hat{\pi}(s) \hat{c}(s) d\mu(s) = \int_S \hat{\pi}(s) e^r(s) d\mu(s).$$

For all q in $\Delta(T)$,

$$\hat{\pi}(q) = \int_S \hat{\pi}(s) dq^{[0,1]}.$$

Without loss of generality, $\hat{\pi}$ depends on at most countably many $r \in [0, 1]$. Therefore, for all n , for λ -a.e. $r \in I_n$ — those r on which $\hat{\pi}$ does not depend,

$$\begin{aligned} \int_S \hat{\pi}(s) e^r(s) d\mu(s) &= \int_{\Delta(T)} \left[\int_S \hat{\pi}(s) e^r(s) dq^{[0,1]} \right] d\nu(q) = \\ \int_{\Delta(T)} \left[\int_S \hat{\pi}(s) dq^{[0,1]} \int_S e^r(s) dq^{[0,1]} \right] d\nu(q) &= \int_{\Delta(T)} \hat{\pi}(q) \left[\sum_t e_t^n q(t) \right] d\nu(q). \end{aligned}$$

The budget constraint for these r then reduces to

$$\int_{\Delta(T)} \hat{\pi}(q) \hat{x}^r(q) d\nu(q) = \int_{\Delta(T)} \hat{\pi}(q) \left[\sum_t e_t^n q(t) \right] d\nu(q).$$

For $q \in \Delta(T)$, $\hat{p}(\chi_q) = \hat{\pi}(q)$, and $\hat{p}(e^r) = \int_S \hat{\pi}(s) e^r(s) d\mu(s)$.

The portfolio \hat{m}^r for individual $r \in I_n$ is constructed by putting weight -1 on e^r and $+1$ on the “spread” over the χ'_q ’s defined by

$$\hat{m}^r(B) = \int_B \hat{x}^r(q) d\nu(q) \quad \text{for all measurable } B \subset \Delta(T).$$

The portfolio \hat{m}^r goes short the endowment of individual r and long each χ_q with density $\hat{x}^r(q)$, according to ν . This portfolio generates \hat{c}^r , and costs nothing by no - arbitrage. The q -average endowments of individuals in I_n is $\bar{e}^n(q) = \sum_t e_t^n q(t)$. The function $q \rightarrow \bar{e}^n(q)$ is continuous and therefore measurable on $\Delta(T)$. The asset $\sum_t e_t^n f_t$ can be generated by a portfolio, \tilde{m} , of the elementary securities χ_q defined by

$$\tilde{m}(B) = \int_B \bar{e}^n(q) d\nu(q) \quad \text{for all measurable } B \subset \Delta(T).$$

From the definition of no - arbitrage and, since, by definition, $\hat{p}(\chi_q) = \hat{\pi}(q)$ it follows that

$$\hat{p}\left(\sum_t e_t^n f_t\right) = \hat{p}(\tilde{m}) = \int_{\Delta(T)} \hat{\pi}(q) \left(\sum_t e_t^n q(t)\right) d\nu(q).$$

Consequently, for λ - a.e. $r \in I_n$,

$$\hat{p}(e^r) = \hat{p}\left(\sum_t e_t^n f_t\right) = \int_{\Delta(T)} \hat{\pi}(q) \left(\sum_t e_t^n q(t)\right) d\nu(q).$$

But then, from the budget equation,

$$\hat{p}(e^r) = \int_{\Delta(T)} \hat{\pi}(q) \hat{c}^r(q) d\nu(q)$$

or

$$\hat{p}(\hat{m}^r) = 0.$$

The portfolio \hat{m}^r is an optimal choice at prices \hat{p} if, whenever the individual can obtain c^r at prices \hat{p} , then c^r is in the budget set at the prices $\hat{\pi}$. The individual can obtain c^r at prices \hat{p} if there exists a portfolio, m , such that

$$\hat{p}(m) = 0$$

and

$$c^r(s) = e^r(s) + \int_0^1 e_q^r(s) d_q^* m + \int_{\Delta(T)} \chi_q(s) dm,$$

with the second constraint satisfied $q^{[0,1]}$ - almost everywhere, for every $q \in \Delta(T)$. The value of c^r at the implicit prices for contingent commodities $\hat{\pi}$ is

$$\begin{aligned} & \int_S \hat{\pi}(s) c^r(s) d\mu(s) = \\ & \int_S \hat{\pi}(s) e^r(s) d\mu(s) + \int_S \hat{\pi}(s) \left[\int_0^1 e_q^r(s) d_q^* m \right] d\mu(s) + \\ & \int_S \hat{\pi}(s) \left[\int_{\Delta(T)} \chi_q(s) dm \right] d\mu(s) = \\ & \int_S \hat{\pi}(s) e^r(s) d\mu(s) + \int_0^1 \left[\int_S \hat{\pi}(s) e_q^r(s) d\mu(s) \right] dm + \\ & \int_{\Delta(T)} \left[\int_S \hat{\pi}(s) \chi_q(s) d\mu(s) \right] dm = \\ & \int_S \hat{\pi}(s) e^r(s) d\mu(s) + \int_0^1 \hat{P}(e^r) dm + \int_{\Delta(T)} \hat{\pi}(q) dm = \\ & \int_S \hat{\pi}(s) e^r(s) d\mu(s), \end{aligned}$$

where the last equality uses the fact that, by no - arbitrage, $\hat{p}(m) = \int_0^1 \hat{p}(e^r) dm + \int_{\Delta(T)} \hat{\pi}(q) d$. \square

The optimal allocation of risks can thus be obtained by means of as many mutual funds as there are classes of individuals, and as many elementary securities as there are aggregate states.

Remark The proposition above implies that if (\hat{c}, \hat{p}) is a competitive equilibrium with implicit prices for contingent commodities and a single, over - all budget constraint, then it is in fact an equilibrium of the economy with the primary assets A . To make this statement precise, however, one should introduce a non - profit agency that would supply the elementary securities χ_q ; these non - essential details are avoided in the presentation above.

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