Prices Versus Quantities in a Vintage Capital Model

Thierry Bréchet, Tsvetomir Tsachev, and Vladimir M. Veliov

Abstract The heterogeneity of the available physical capital with respect to productivity and emission intensity is an important factor for policy design, especially in the presence of emission restrictions. In a vintage capital model, reducing pollution requires to change the capital structure through investment in cleaner machines and to scrap the more polluting ones. In such a setting we show that quantity-based or a price-based regulation may yield contrasting outcomes. We also show that some failures in the permits market may undermine its efficiency and that imposing the emission cap over longer periods plays a regularizing role in the market, that is, ensures a positive market price of permits and decreases its volatility.

1 Introduction

It is well-established in the economic literature that regulating pollution through prices (e.g. emission charges) or quantities (e.g. emission quotas) is equivalent. Both yield the same resource allocation and welfare level. In his seminal paper Weitzman (1974) showed that such equivalence does not hold anymore when information is imperfect, be it on pollution abatement costs or damages. Following Weitzman, many papers elaborated on the uncertainty issue and the choice of policy instrument (for some recent papers, see Zhao 2003; Krysiac 2008). A few authors introduced alternative motives for which this equivalence may not hold. As examples, Finkelshtain and Kislev (1997) found political motives, and Kelly (2005) stressed the role of general equilibrium effects. The contribution of our paper is to show that this equivalence may not hold even under perfect information, simply because the capital stock that generates pollution is not homogeneous. In a vintage capital model, reducing pollution requires to change the capital structure through investment in cleaner machines and scrapping the more polluting ones. In such a setting we show that emission tax and auction emission permits may yield contrasting outcomes.

This research was partly financed by the Austrian Science Foundation (FWF) under grant No. P18161-N13 and by the Belgian Science Policy under the CLIMNEG project (SD/CP/05A).

T. Bréchet (✉)
CORE, and Chair Lhoist Berghmans in Environmental Economics and Management, Université catholique de Louvain, Voie du Roman Pays, 34, 1348 Louvain-la-Neuve, Belgium
e-mail: thierry.brechet@uclouvain.be

J.C. Cuaresma et al. (eds.), *Dynamic Systems, Economic Growth, and the Environment,* Dynamic Modeling and Econometrics in Economics and Finance 12,
DOI 10.1007/978-3-642-02132-9_7,

The heterogeneity of the available physical capital with respect to productivity and emission intensity is an important factor for policy design, especially in the presence of emission restrictions. To decide which machines to scrap and which machines to buy is an indispensable right of the firm's management. The vintage model of a firm that we employ is essentially a version of the one introduced in Barucci and Gozzi (1998, 2001) and investigated by several authors (see also Feichtinger et al. (2006, 2008) and the bibliography therein). In particular, the model allows for investing and scrapping in technologies of any vintage.[1] Polluting emissions are regulated either by a tax or with auctioned pollution permits.

When the price of emission is endogenized by auctioning tradable emission permits (and not set as a tax) it has to be determined by a market equilibrium equation that involves the emission paths resulting from the optimal behavior of the participating firms. The derivation and the investigation of this equation is the main goal of this paper. It turns out that the equation for the auction price of emissions (being a Fredholm integral equation of the first kind) is ill-posed, in the sense that (i) it may fail to have a solution; (ii) it may have multiple solutions; (iii) it may not have a positive solution (in which case the solution does not represent a market price); (iv) the solution, even it exists and is unique and positive, may be highly volatile and fluctuating. For this reason we introduce period-wise restrictions on the emission, which correspond to the *commitment periods* in the terminology of the Kyoto protocol. It is argued below that such a period-wise emission restriction plays a regularizing role on the auction price of emissions. Namely, a sufficiently large commitment period ensures existence, uniqueness (under somewhat restrictive conditions) and (according to numerical evidence) positiveness of the solution of the auction price equation. In addition it decreases the volatility of the auction market.

The paper is organized as follows. In Sect. 2 we present the basic model of a firm facing an exogenous cost for emissions (a tax) and we characterize its optimal behavior. The model is of vintage-type, that is, the physical capital is differentiated with respect to technologies of different dates. In Sect. 3 we present some properties and analytic expressions of the emission of the firm along its optimal path as a function of the exogenous costs. Based on this, in Sect. 4 we investigate the equilibrium equation for the auction of emission permits and introduce period-wise emission restrictions (commitment periods) as a tool for regularization of the auction price. Then in Sect. 5 we present some numerical results supporting the regularizing role of the period-wise emission restrictions.

2 The Firm's Problem with an Emission Tax

The model of the firm presented below is a version of the PDE-vintage models introduced in Barucci and Gozzi (1998, 2001) and further investigated and applied

---

[1] This is a substantial difference with the delay-differential equation models, see Boucekkine et al. (2004, 2005).

in a large number of contributions (see Feichtinger et al. 2006, 2008 and the literature therein). The formal difference of our model with the abovementioned ones is technical and not essential from a methodological point of view.

First we describe the basic model of a firm that is composed of machines of different vintages (technologies) τ: $K(\tau, s)$ will denote the capital stock of vintage τ and of age s. That is, $K(\tau, t-\tau)$ is the stock of vintage τ that exists at time $t \geq \tau$. The maximal life-time of machines of each technology will be denoted by ω, and the depreciation rate of each technology—by δ. Both are assumed independent of the vintage just for notational convenience. At any time $t > 0$ the firm may invest with intensity $I(\tau, s)$ in machines of vintage τ that are of age s at time t (so that $s = t - \tau$).

The planning horizon of the firm is $[0, \infty)$, therefore the stock of machines of vintage $\tau \in [-\omega, 0]$ which are present in the firm at time $t = 0$ is considered as exogenous and will be denoted by $K_0(\tau)$. These machines have age $-\tau$ at time $t = 0$ and may be in use until they reach age ω. Machines of vintage $\tau > 0$ may be in use for ages $s \in [0, \omega]$, and their stock at age zero equals zero. Therefore $K_0(\tau)$ will be defined as zero for $\tau > 0$. The ages of possible use of any vintage can be written as $[s_0(\tau), \omega]$, where $s_0(\tau) = \max\{0, -\tau\}$.

Summarizing, the dynamics of each vintage $\tau \in [-\omega, \infty)$ is given by the equation

$$\dot{K}(\tau, s) = -\delta K(\tau, s) + I(\tau, s), \quad K(\tau, s_0(\tau)) = K_0(\tau), \quad s \in [s_0(\tau), \omega], \tag{1}$$

where "dot" means everywhere the derivative with respect to s (the argument representing the age).

The productivity of machines of vintage τ is denoted by $f(\tau)$, while $g(\tau)$ denotes the emission per machine of vintage τ. The firm faces costs due to emissions at an exogenous price $v(t)$ per unit of emission. At this stage of the paper $v(t)$ represents a tax on pollution set up by a regulator. This price will be endogenized later on when auctioned emission permits will be considered. Due to this cost the firm may decide to (possibly temporarily) switch off a part of the machines. Let us denote by $W(\tau, s) \in [0, 1]$ the fraction of the machines of vintage τ that operate at age s.

The cost of investment I in s years old machines of any vintage will be denoted by $C(s, I)$.

The present value (at time $t = 0$) of the total production of machines of vintage τ, discounted with a rate r, is

$$e^{-r\tau} \int_{s_0(\tau)}^{\omega} e^{-rs} f(\tau) K(\tau, s) W(\tau, s) ds,$$

the cost of emission is

$$e^{-r\tau} \int_{s_0(\tau)}^{\omega} e^{-rs} v(\tau + s) g(\tau) K(\tau, s) W(\tau, s) ds,$$

and the investment costs are

$$e^{-r\tau} \int_{s_0(\tau)}^{\omega} e^{-rs} C(s, I(\tau, s)) ds.$$

The firm maximizes the aggregated over time discounted net revenue, that is, solves the problem

$$\max_{I \geq 0, W \in [0,1]} \int_{-\omega}^{\infty} e^{-r\tau} \int_{s_0(\tau)}^{\omega} e^{-rs} [(f(\tau) - v(\tau + s) g(\tau)) K(\tau, s) W(\tau, s) - C(s, I(\tau, s))] ds d\tau \tag{2}$$

subject to (1).

The emission of the firm at time $t > 0$ is given by the expression

$$E(t) = \int_{t-\omega}^{t} g(\tau) K(\tau, t - \tau) W(\tau, t - \tau) d\tau. \tag{3}$$

Remark 1 Due to the specific form of the problem there is an easy way to define optimality even though the optimal value in (2) may be infinite. Namely (I^*, W^*, K^*) is a solution of (2), (1) if for any $T > 0$ the restriction of these functions to $D_T = \{(\tau, s) : \tau \in [-\omega, T],\ s \in [s_0(\tau), \omega]\}$ is an optimal solution of the problem in which the integration in (2) is carried out on D_T.

Problem (2), (1) fits to the general framework of heterogeneous optimal control systems developed in Veliov (2008). However, the present problem can be treated also by the classical Pontryagin optimality conditions for ODEs since it decomposes along vintages: every technology vintage $\tau \in [-\omega, \infty)$ is managed separately by solving the problem

$$\max_{i \geq 0, w \in [0,1]} \int_{s_0(\tau)}^{\omega} e^{-rs} [(f(\tau) - v(\tau + s) g(\tau)) k(s) w(s) - C(s, i(s))] ds \tag{4}$$

subject to

$$\dot{k}(s) = -\delta k(s) + i(s), \quad k(s_0(\tau)) = K_0(\tau), \quad s \in [s_0(\tau), \omega]. \tag{5}$$

If for any fixed $\tau \in [-\omega, \infty)$ the triple $(i(\cdot), w(\cdot), k(\cdot)) = (I(\tau, \cdot), W(\tau, \cdot), K(\tau, \cdot))$ is a solution of this problem, then (I, W, K) is a solution of (2), (1) and vice versa.[2]

The following is assumed throughout the paper.

(A1) The exogenous data K_0, f, g are non-negative and continuous, f and g are continuously differentiable, $g > 0$, $f' > 0$, $g' \leq 0$, $r \geq 0$, $\delta \geq 0$;

[2] This is not a self-evident fact, but can easily be proven in natural space settings for the two problems and on the assumptions made below.

(A2) The cost function $C(s,i)$ is two times differentiable in i, the derivatives C_i' and C_{ii}'' are continuous in (s,i), $C(s,0)=0$, $C_i'(s,0)\ge 0$, $C_{ii}''\ge \varepsilon_C>0$;
(A3) The price of emission, $v(\cdot)$, is a measurable bounded function.[3]

Assumption (A1) means that newer machines are more productive and less polluting than older machines Under these conditions problem (2), (1) has a unique solution $(I^*[v], W^*[v], K^*[v])$. The corresponding emission given by (3) will be denoted by $E^*[v]$.

Since W enters only in the objective function, clearly we have

$$W^*[v](\tau,s)=\begin{cases}0 & \text{if } f(\tau)-v(\tau+s)g(\tau)\le 0,\\ 1 & \text{if } f(\tau)-v(\tau+s)g(\tau)>0.\end{cases} \tag{6}$$

The optimal control i^* of problem (4), (5) for a fixed τ and $v(\cdot)$ is easy to obtain by applying the Pontryagin maximum principle (Pontryagin et al. 1962). Namely,

$$i^*(s)=\begin{cases}0 & \text{if } \xi^*(s)<C_i'(s,0),\\ (C_i')^{-1}(s,\xi^*(s)) & \text{if } \xi^*(s)\ge C_i'(s,0),\end{cases} \tag{7}$$

where $\xi\to(C_i')^{-1}(s,\xi)$ is the inverse of the function $i\to C_i'(s,i)$ and $\xi^*(s)$ is the unique solution of the *adjoint equation*

$$\dot{\xi}(s)=(r+\delta)\xi(s)-(f(\tau)-v(\tau+s)g(\tau))w^*(s), \qquad \xi(\omega)=0. \tag{8}$$

3 The Emission Mapping

An exogenously given tax on emission, that is a function $v(t)$ as in assumption (A3), determines the optimal emission of the firm

$$E^*[v](t)=\int_{t-\omega}^{t} g(\tau)K^*[v](\tau,t-\tau)W^*[v](\tau,t-\tau)d\tau, \tag{9}$$

where $(I^*[v], W^*[v], I^*[v])$ is the optimal solution at the firm level corresponding to v. In this section we investigate in some more details the mapping $v\to E^*[v]$.

Since the emission restriction takes effect at time $\hat{t}$, we consider v as equal to zero before $\hat{t}$. For a technical convenience we assume that $\hat{t}\ge\omega$, although this is not essential.

Denote by $\mathcal{V}$ the space of all measurable and locally bounded functions $v:[0,\infty)\mapsto \mathbf{R}$ that equal zero on $[0,\hat{t})$. The space $\mathcal{V}$ will be sometimes considered as a subspace of $L_\infty^{\text{loc}}(0,\infty)$, the latter consisting of all measurable function that are bounded on every bounded interval $[0,T]$.

[3] For reasons that will become clear later we formally allow for negative values of $v(t)$.

The next proposition claims a specific Lipschitz continuity property of E^*. Since we allow for negative values of v we denote $v_- = \min\{0, v\}$. In the text below we shall use also the notation

$$\sigma(\tau, v) = f(\tau) - vg(\tau).$$

Proposition 1 *Assume that* (A1)–(A3) *hold. There is a constant c and a non-increasing function* $[0, \infty) \ni t \to \gamma_t > 0$ *such that for every* $v', v'' \in \mathcal{V}$ *and* $t > \hat{t}$

$$|E^*[v'](t) - E^*[v''](t)| \le c\Bigg[\|v' - v''\|_{L_\infty[t-\omega,t+\omega]} + \frac{f(t+\omega) + \|v'_- + v''_-\|_{L_\infty[t-\omega,t+\omega]}}{\gamma_t}|v'(t) - v''(t)|\Bigg].$$

Proof The function γ_t can be explicitly defined as $\gamma_t = \inf_{\tau \le t} \gamma(\tau) > 0$, where $\gamma(\tau) = f'(\tau) - \frac{f(-\omega)}{g(-\omega)} g'(\tau)$, which is strictly positive according to (A1).

For arbitrarily fixed $v', v'' \in \mathcal{V}$ denote $\Delta K = K^*[v'] - K^*[v'']$, $\Delta I = I^*[v'] - I^*[v'']$, $\Delta W = W^*[v'] - W^*[v'']$, $\Delta\xi = \xi^*[v'] - \xi^*[v'']$. We have

$$|E^*[v'](t) - E^*[v''](t)| \le \int_{t-\omega}^{t} g(\tau)|\Delta K(\tau, t-\tau)|W^*[v'](\tau, t-\tau)d\tau + \int_{t-\omega}^{t} g(\tau)K^*[v''](\tau, t-\tau)|\Delta W(\tau, t-\tau)|d\tau. \tag{10}$$

Using (6) we obtain from the adjoint equation (8), which is satisfied by $\xi[v_k](s)$ for (almost) every τ, that

$$\begin{aligned}|\Delta\xi(\tau, s)| &\le \int_s^{\omega} e^{-(r+\delta)(\theta-s)}|\chi(f(\tau) - v'(\tau+\theta)g(\tau)) \\ &\quad - \chi(f(\tau) - v''(\tau+\theta)g(\tau))|d\theta \\ &\le \omega\bar{g}\|v' - v''\|_{L_\infty[\tau,\tau+\omega]},\end{aligned}$$

where $\chi(x) = x$ for $x > 0$, $\chi(x) = 0$ for $x \le 0$, $\bar{g} = g(-\omega)$ is an upper bound for g, and we use that χ is Lipschitz continuous with Lipschitz constant equal to one.

From assumption (A2) we easily obtain that $(C_i')^{-1}(s, \cdot)$ is Lipschitz continuous with Lipschitz constant $1/\varepsilon_C$, thus according to (7)

$$|\Delta I(\tau, s)| \le \frac{\omega\bar{g}}{\varepsilon_C}\|v' - v''\|_{L_\infty[\tau,\tau+\omega]}.$$

Then from (1) or (5) we obtain that

$$|\Delta K(\tau, s)| \le \frac{\omega^2\bar{g}}{\varepsilon_C}\|v' - v''\|_{L_\infty[\tau,\tau+\omega]}, \quad s \in [s_0(\tau), \omega].$$

Thus the first term in (10) can be estimated by

$$\frac{\omega^3 \bar{g}^2}{\varepsilon_C} \|v' - v''\|_{L_\infty[t-\omega,t+\omega]}. \tag{11}$$

To estimate the second term in (10) we first note that by a similar argument as above we estimate

$$\begin{aligned}\xi[v''](\tau,s) &\le \int_s^\omega e^{-(r+\delta)(\theta-s)} \chi(f(\tau) - v''(\tau+\theta)g(\tau))d\theta \\ &\le \omega\big(f(\tau) + \bar{g}\|v''_-\|_{L_\infty[\tau,\tau+\omega]}\big).\end{aligned}$$

Using that the function $(C_i')^{-1}(s,\cdot)$ is Lipschitz continuous with Lipschitz constant $\le 1/\varepsilon_C$, which is implied in a standard way by (A2), and (7) we obtain that

$$I[v''](\tau,s) \le \frac{\omega}{\varepsilon_C}\big(f(\tau) + \bar{g}\|v''_-\|_{L_\infty[\tau,\tau+\omega]}\big).$$

Hence, using that $t \ge \hat{t} \ge \omega$,

$$K[v''](\tau,s) \le \frac{\omega^2}{\varepsilon_C}(f(\tau) + \bar{g}\|v''_-\|_{L_\infty[\tau,\tau+\omega]}). \tag{12}$$

It remains to estimate the term

$$|\Delta W(\tau, t-\tau)| = \begin{cases} 0 & \text{if } \sigma(\tau, v'(t)) \text{ and } \sigma(\tau, v''(t)) \text{ are both positive} \\ & \text{or both non-positive,} \\ 1 & \text{otherwise} \end{cases}$$

in (10). To do this we compare the functions $\sigma(\tau, v'(t))$ and $\sigma(\tau, v''(t))$, assuming without any restriction that $v'(t) \le v''(t)$. Clearly $\Delta W(\tau, t-\tau) \ne 0$ (and equals one) for some $\tau \in [t-\omega, t]$ if and only if $\sigma(\tau, v''(t)) \le 0 < \sigma(\tau, v'(t))$.

Note also that if $v < 0$, then $\sigma(\tau, v)$ and $\sigma(\tau, 0)$ have both positive signs, hence the sign of $\sigma(\tau, v)$ does not change if we replace v with $v_+ := \max\{0, v\}$. Since $|v'_+ - v''_+| \le |v' - v''|$, the estimations below would not change if we replace $v'(t)$ and $v''(t)$ with $v'(t)_+$ and $v''(t)_+$, or equivalently, if we assume that $v'(t) \ge 0$.

Due to (A1) (implying that $\sigma(\tau, v)$ is strictly increasing for $v \ge 0$) and (A3) the set

$$\{\tau \in [t-\omega, t] : \sigma(\tau, v''(t)) \le 0 < \sigma(\tau, v'(t))\} \tag{13}$$

(if nonempty) is an interval $(\tilde{\tau} - \nu, \tilde{\tau}]$, where $\sigma(\tilde{\tau}, v''(t)) = 0$ and $\sigma(\tilde{\tau} - \nu, v'(t)) \ge 0$. Then

$$\sigma(\tilde{\tau} - \nu, v''(t)) = \sigma(\tilde{\tau}, v''(t)) - \dot{\sigma}(\tilde{\tilde{\tau}}, v''(t))\nu = -\dot{\sigma}(\tilde{\tilde{\tau}}, v''(t))\nu,$$

where $\tilde{\tilde{\tau}} \in [\tilde{\tau} - \nu, \tilde{\tau}]$. Subtracting the above equality from $\sigma(\tilde{\tau} - \nu, v'(t)) \ge 0$ we obtain that

$$(v''(t) - v'(t))g(\tilde{\tau} - \nu) \ge \dot{\sigma}(\tilde{\tilde{\tau}}, v''(t))\nu = \big(f'(\tilde{\tilde{\tau}}) - v''(t)g'(\tilde{\tilde{\tau}})\big)\nu. \tag{14}$$

Since

$$0 \geq \sigma(\tilde{\tilde{\tau}}, v''(t)) = f(\tilde{\tilde{\tau}}) - v''(t)g(\tilde{\tilde{\tau}}),$$

we have according to (A1) that

$$v''(t) \geq \frac{f(\tilde{\tilde{\tau}})}{g(\tilde{\tilde{\tau}})} \geq \frac{f(-\omega)}{g(-\omega)}.$$

Then (14) and $g'(\tilde{\tilde{\tau}}) \leq 0$ imply

$$(v''(t) - v'(t))g(\tilde{\tau} - \nu) \geq \left(f'(\tilde{\tilde{\tau}}) - \frac{f(-\omega)}{g(-\omega)}g'(\tilde{\tilde{\tau}})\right)\nu = \gamma(\tilde{\tilde{\tau}})\nu \geq \gamma_t \nu.$$

Hence

$$\nu \leq \frac{\bar{g}}{\gamma_t}(v''(t) - v'(t)).$$

Thus the measure of the set in (13) does not exceed $\frac{\bar{g}}{\gamma_t}|v''(t) - v'(t)|$. Combining this with (12), (11) and (10) we obtain the claim of the proposition. □

Lemma 1 *For every $v \in \mathcal{V}$ and $t \geq \hat{t}$ the value $E^*[v](t)$ depends only on the restriction of v to $[t - \omega, t + \omega]$.*

Proof The proof of this lemma is straightforward: $E^*[v](t)$ depends only on $K^*[v](\tau, t - \tau)$ for $\tau \in [t - \omega, t]$ (see (9)), $K^*[v](\tau, t - \tau)$ depends only on $I^*[v](\tau, s)$ with $s \in [0, \omega]$, the latter depends only on $\xi^*[v](\tau, s)$, which on its turn, depends on $v(\theta)$ with $\theta \in [\tau, \tau + \omega]$ (see (8)). □

The analysis of the market price of emissions in the next section involves a rather complicated functional equation for v. Together with the general case we consider also a particular cost function $C(s, I)$ for which the equation substantially simplifies.

(A4) $C(s, I) = 1/2c(s)I^2$, where $c(s) > 0$, $s \in [0, \omega]$, is a measurable non-increasing function.

The advantage of the above assumption, which is still economically meaningful, is that the optimal investment becomes a linear function of the "shadow price" of the capital stock. Namely, (7) becomes

$$I^*[v](\tau, s) = \frac{\xi^*(\tau, s)}{c(s)}, \quad s \in [0, \omega), \tag{15}$$

where $\xi(\tau, \cdot)$ is the solution of the adjoint equation (8), that is,

$$\dot{\xi}(\tau, s) = (r + \delta)\xi(\tau, s) - \sigma(\tau, v(\tau + s))W^*[v](\tau, s), \qquad \xi(\tau, \omega) = 0. \tag{16}$$

Note that due to (A1) the function $\tau \longrightarrow \sigma(\tau, v)$, restricted to an interval $[t-\omega, t]$, either has a definite sign or has a single zero in this interval, denoted further by $\theta(t, v)$. In addition, we set $\theta(t, v) = t - \omega$ or $\theta(t, v) = t$ if $\sigma(\tau, v) > 0$ or $\sigma(\tau, v) < 0$ in $[t-\omega, t]$, respectively. With this notation we have, according to (6), that $W^*[v](\tau, t-\tau) = 1$ for $\tau \in (t-\omega, t)$ if and only if $\tau \in (\theta(t, v(t)), t)$.

Lemma 2 *Let assumptions* (A1) *and* (A4) *hold and let* $v \in \mathcal{V}$. *Then for every* $t \geq \hat{t}$

$$E^*[v](t) = \int_{\theta(t,v(t))}^{t} \int_{\theta(t,v(t)) \vee \theta(\alpha,v(\alpha))}^{\alpha} g(\tau)\kappa(t,\tau,\alpha-\tau)\sigma(\tau, v(\alpha))d\tau\, d\alpha$$
$$+ \int_{t}^{\theta(t,v(t))+\omega} e^{-(r+\delta)(\alpha-t)} \int_{\theta(t,v(t)) \vee \theta(\alpha,v(\alpha))}^{t} g(\tau)\kappa(t,\tau,t-\tau)$$
$$\times \sigma(\tau, v(\alpha))d\tau\, d\alpha$$
$$+ \int_{\theta(t,v(t))+\omega}^{t+\omega} e^{-(r+\delta)(\alpha-t)} \int_{\theta(\alpha,v(\alpha))}^{t} g(\tau)\kappa(t,\tau,t-\tau)\sigma(\tau, v(\alpha))d\tau\, d\alpha,$$

where $a \vee b := \max\{a, b\}$ *and*

$$\kappa(t, \tau, \beta) = \int_0^\beta \frac{1}{c(s)} e^{-\delta(t-\tau-s)-(r+\delta)(\beta-s)} ds.$$

The proof of this lemma uses the Cauchy formula for the solution ξ of the adjoint equation (16), the formula (15) for the optimal control, the Cauchy formula for the corresponding solution $K^*[v]$ of (1), and (9). This results in an expression for $E^*[v](t)$ in the form of a triple integral depending only on the data and v, from which one can derive the expression in the lemma after a sequence of changes of the order and the variables of integration. We skip the cumbersome calculations.

Definition We call the price function $v \in \mathcal{V}$ *regular* if $f(\tau) - v(t)g(\tau) > 0$ for all $\tau \geq 0$ and $t \in (\tau, \tau + \omega)$.

In other words, regularity means that the price $v(t)$ does not invoke switch-off of existing machines. In the context of the emission restrictions the existence of a regular auction price (see next section) would mean that the environmental goals can be achieved only by appropriate investment policies (without premature scrapping).

For a regular $v \in V$ the expression for $E^*[v]$ substantially simplifies, since $\theta(t, v(t)) = t - \omega$ for all $t \geq \hat{t}$. Hence,

$$E^*[v](t) = \int_{t-\omega}^{t} \int_{t-\omega}^{\alpha} g(\tau)\kappa(t,\tau,\alpha-\tau)\sigma(\tau, v(\alpha))d\tau\, d\alpha$$
$$+ \int_{t}^{t+\omega} e^{-(r+\delta)(\alpha-t)} \int_{\alpha-\omega}^{t} g(\tau)\kappa(t,\tau,t-\tau)\sigma(\tau, v(\alpha))d\tau\, d\alpha.$$

Having in mind the definition of $\sigma(\tau, v)$ we split each of the above outer integrals into two parts: one term depending on v, the other—independent of v. The resulting expression that is independent of v is exactly the emission corresponding to $v(t) \equiv 0$. Having in mind also that $v(t) = 0$ for $t < \hat{t}$ we obtain the following lemma.

Lemma 3 *Let assumptions* (A1) *and* (A4) *hold and let* $v \in \mathcal{V}$ *be regular. Then for every* $t \geq \hat{t}$

$$E^*[v](t) = E^*[0](t) - \int_{\hat{t} \vee (t-\omega)}^{t+\omega} \varphi(t, \alpha) v(\alpha) d\alpha, \tag{17}$$

where

$$\varphi(t, \alpha) = \begin{cases} \int_{t-\omega}^{\alpha} g^2(\tau) \kappa(t, \tau, \alpha - \tau) d\tau & \text{if } \alpha \in [t - \omega, t], \\ e^{-(r+\delta)(\alpha - t)} \int_{\alpha-\omega}^{t} g^2(\tau) \kappa(t, \tau, t - \tau) d\tau & \text{if } \alpha \in [t, t + \omega] \end{cases}$$

and $a \vee b = \max\{a, b\}$, $a \wedge b = \min\{a, b\}$.

The following properties of the kernel $\varphi(t, \cdot)$ play a role in the study of the auction price of emissions defined in the next section.

Lemma 4 *Let assumptions* (A1) *and* (A4) *hold. Then for every* $t > \hat{t}$ *the function* $\varphi(t, \cdot) : [t - \omega, t + \omega]$ *has the following properties*:

$$\varphi(t, t - \omega) = \varphi(t, t + \omega) = \varphi'_\alpha(t, t - \omega) = \varphi'_\alpha(t, t + \omega) = 0,$$

$\varphi'_\alpha(t, \alpha)$ *is strictly positive* (*negative*) *on* $(t - \omega, t)$ (*on* $(t, t + \omega)$, *respectively*).

That is, the "mass" of the kernel $\varphi(t, \cdot)$ is concentrated mostly around $\alpha = t$.

4 The Auction Price of Permits and Its Regularization

We consider an economy consisting of n identical firms described by the model considered in Sect. 2. Let $\hat{E}(t)$ be a cap for the emission permits of the economy for $t \geq \hat{t}$. As before we assume that the regulation (here, the emission cap) is known by the firms at time $t = 0$ for all the future. The cap takes effect at time $\hat{t} \geq \omega$.

The emission permits are auctioned at time $\hat{t}$. The main issue investigated in this section is: does the auction (the primary market) really determine the price $v(t)$? Since the answer is "NO", we analyze theoretically and numerically the primary market behavior and the appearance of market failure.

The equation for the auction price of emission permits, $v(t)$, is

$$E^*[v](t) = \hat{E}(t)/n, \quad t \geq \hat{t}. \tag{18}$$

The market would determine the price of permits if

(i) equation (18) has a solution $v(\cdot) \in \mathcal{V}$;
(ii) the solution is unique;
(iii) the solution is positive for all t.

In general, neither of the above requirement is fulfilled. The first needs assumptions for the data, and these assumptions are difficult to specify due to the complexity of (18) (even in the special case considered in Lemma 3). The second requirement is obviously not fulfilled if, for example, $\hat{E}(t) \equiv 0$, since if v is a solution of (18), then every $\tilde{v}$ with $\tilde{v}(t) > v(t)$ would also be a solution. Even if the above two possibilities could be classified as “academic”, as we show below, a failure of the third requirement could be rather realistic.

Problems for which a meaningful solution may fail to exist (in the present case—a positive market price) or the solution is not unique (indeterminacy), or a unique solution exists but is arbitrarily sensitive to perturbations, are usually called *ill-posed*. A modification (approximation) of an ill-posed problem that turns it into a well posed one is called *regularization*. Regularization methods for equations of the kind we face in (18) are developed first by A. Tikhonov in the 60-th years of the last century. The regularization that we employ below is different and has the advantage that it has a clear economic meaning and is implementable in the real market (see the next two paragraphs for further explanations).

In the simplest case, in which the price function v solving (18) is regular, it solves also the equation

$$\int_{\hat{t}\vee(t-\omega)}^{t+\omega} \varphi(t,\alpha)v(\alpha)d\alpha = E^*[0](t) - \hat{E}(t) \tag{19}$$

(we skip the division by n considering $\hat{E}(t)$ as already divided by n). This is a Fredholm integral equation of the first kind on the infinite interval $[\hat{t}, +\infty)$. Such equations are inherently ill-posed. To see that it is enough to add to a solution v a highly fluctuating term, such as $\sin(nt)$. Then $v(t) + \sin(nt)$ would be a solution of the same equation with the right-hand side modified by the quantity $\int_{\hat{t}\vee(t-\omega)}^{t+\omega} \varphi(t,\alpha)\sin(n\alpha)d\alpha$, which is arbitrarily small when n is large enough. Thus an arbitrarily small disturbance in the right-hand side can lead to a finite (even arbitrarily large) change of the solution. This difficulty is caused by the smoothing effect on v of the integration with φ: high-frequency components of v are “smoothed out”. Therefore one can expect that computing v would tend to amplify any high-frequency component or irregularity of the right-hand side. In effect, the right-hand side of (19) has to be somewhat “smoother” than the solution v in order to obtain satisfactory numerical approximation (Hansen 1992).

A good numerical method to solve a Fredholm integral equation of the first kind should be able to somehow filter out the high-frequency components in the singular value expansion of the solution (if such exists). Different methods of regularization aimed at finding reasonable numerical approximation to the solution are known (Tikhonov and Arsenin 1977; Delves and Mohamed, 1985; Kress 1989). However, we stress that our aim is not just to solve the price equation (19) or some of its more complicated nonlinear versions (say, that resulting from

Lemma 2). Our ultimate goal is to imitate the behavior of the auction market, therefore the regularization we apply for solving (18) should be implementable also in the real auction market. We argue below and in the next section that a relevant regularization mean is to formulate the emission constraint (18) period-wise, rather than at any time instant. This amounts to applying *regularization by time-aggregation*, related to the Nyström's method (Delves and Mohamed, 1985, Chap. 12). In contrast to the celebrated singular expansion/decomposition this method is directly applicable also in the general (nonlinear) case of (18) and has a clear policy implementation.

Namely, we introduce the discrete version of the emission mapping E^* as follows. Assume that an emission restriction is given period-wise: $\hat{E}_k = \frac{1}{t_k - t_{k-1}} \times \int_{t_{k-1}}^{t_k} \hat{E}(t)dt$ is the cap for the emission intensity in the time-period $[t_{k-1}, t_k]$. For simplicity we assume that all time periods have the same length $h > 0$, thus $t_k = kh$, and also that $\omega = mh$, $\hat{t} = \hat{k}h$ for appropriate natural numbers m and $\hat{k}$. Correspondingly, the price of emission will be constant, v_k, on each interval $[t_{k-1}, t_k)$ and zero for $k \le \hat{k}$. Thus instead of the space $\mathcal{V}$ of price functions we consider

$$\mathcal{V}^h = \{(v_1, v_2, \ldots) : v_k \in (-\infty, +\infty),\ v_k = 0 \text{ for } k \le \hat{k}\},$$

which can be viewed as a subset of $\mathcal{V}$ by piece-wise constant embedding of $\mathcal{V}^h$ in $\mathcal{V}$. The emission resulting from $v \in \mathcal{V}^h$ becomes a vector $E^{*h}[v] = (E_1^{*h}[v], E_2^{*h}[v], \ldots)$, where

$$E_k^{*h}[v] = \frac{1}{h} \int_{t_{k-1}}^{t_k} \int_{t-\omega}^{t} g(\tau) K^*[v](\tau, t-\tau) W^*[v](\tau, t-\tau) d\tau\, dt. \tag{20}$$

Lemma 1 can be directly translated to the discrete case. It claims that $E_k^{*h}[v]$ depends only on $v_{k-m}, \ldots, v_{k+m}$.

Then instead of the price equation (18) we consider the equation (skipping again the division by n)

$$E_k^{*h}[v] = \hat{E}_k^h, \quad k \ge \hat{k} \tag{21}$$

for $v \in \mathcal{V}^h$.

Under the conditions of Lemma 3 the period-wise version of the emission as function of $v \in \mathcal{V}^h$ becomes (after a change of the order of integration):

$$\begin{aligned} E_k^{*h}[v] &= E_k^{*h}[0] - \sum_{i=\hat{k}\vee(k-m)}^{k+m} \frac{1}{h} \int_{B_{ki}} \varphi(t, \alpha) dt\, d\alpha v_i \\ &=: E_k^{*h}[0] - \sum_{i=\hat{k}\vee(k-m)}^{k+m} a_{ki}^h v_i, \quad k \ge \hat{k}, \end{aligned} \tag{22}$$

where B_{ki} is the square $[t_i, t_{i+1}] \times [t_{k-1}, t_k]$ in the (α, t)-plane for $i = k-m, \ldots, k+m-2$, while $B_{k,k-m}$ is the triangle with vertices $\{(t_{k-m-1}, t_{k-1}), (t_{k-m}, t_{k-1}), (t_{k-m}, t_k)\}$, and $B_{k,k+m}$ is the triangle with vertices $\{(t_{k+m-1}, t_{k-1}), (t_{k+m-1}, t_k),$

$(t_{k+m}, t_k)\}$. However, the numerical analysis given in the next section, as well as the proposition below apply to the general case of an emission map $E_k^{*h}[v]$ defined in (20).

In the rest of this section we address the issue of existence of a solution to the (nonlinear non-smooth) equation (21) for $v \in \mathcal{V}^h$. First of all we claim a certain Lipschitz continuity property of the mapping E^{*h}, similarly as in Proposition 1.

Proposition 2 *Assume that* (A1)–(A3) *hold. There is a constant c and a function $\gamma_t > 0$* (*the same as in Proposition* 1) *such that for every $v', v'' \in \mathcal{V}^h$ and $k > \hat{k}$*

$$|E_k^{*h}[v'] - E_k^{*h}[v'']| \\ \le c\left[\max_{k-m<i\le k+m} |v_i' - v_i''| + \frac{f(t_{k+m}) + \max_{k-m<i\le k+m} |v_{i-}' + v_{i-}''|}{\gamma_{t_k}} |v_k' - v_k''|\right].$$

The claim of this proposition follows directly from Proposition 1 and (20).

In the existence result presented below we consider the truncated version of (21). That is, given the aggregation time-step $h > 0$ we solve the finite system of equations

$$E_k^{*h}[v] = \hat{E}_k, \quad k = \hat{k} + 1, \dots, M, \tag{23}$$

with respect to $v = (v_{\hat{k}+1}, \dots, v_M)$ where M is presumably a large number, so that $T = Mh$ is also "very large". According to the discrete version of Lemma 1, $E_k^{*h}[v]$ depends only on $v_{k-m}, \dots, v_{k+m}$. The values v_i for $i = 1, \dots, \hat{k}$ are fixed a priori equal to zero, the values $v_{M+1}, \dots, v_{M+m}$ will be considered as fixed parameters. Thus (23) becomes a system of $M - \hat{k}$ equations for the $M - \hat{k}$ unknown variables $v_{\hat{k}+1}, \dots, v_M$.

It as a rather difficult task to prove that the solutions of (23) would converge to a solution of (21) when $M \longrightarrow +\infty$ (this is not simple also in the linear case (22)). The truncation of the time horizon is based on the common belief that the (hypothetical) knowledge of the economic factors thousands of years from now would not essentially influence the behavior of the economic agents in the next 100 years. Still, the plausibility of the truncation is not evident due to the possibly unlimited technological progress (this is clearly exhibited by the requirement (24) in the proof of the proposition below).

Another support for the truncation of the time horizon is provided by our numerical experiments, where the auction price for the next 80 years (to which we restrict the numerical analysis in the next section) remains practically the same when (23) is solved for $T = Mh = 150$ or more years, and also when the parameters $v_{M+1}, \dots, v_{M+m}$ vary in a reasonable range (taking all of them equal to zero is a relevant choice, since it means that no emission cap is posed after time $T = Mh$).

Proposition 3 *Let $M > \hat{k}$ and $\bar{e} > 0$ be arbitrarily fixed. Then there exists $\bar{v}$ such that for every $\hat{E}_k$, $k = \hat{k} + 1, \dots, M$, with $\hat{E}_k \in [0, \bar{e}]$ and any choice of $v_i \in [0, \bar{v}]$ for $i = M + 1, \dots, M + m$ system* (23) *has a solution $v_{\hat{k}+1}, \dots, v_M \in [-\bar{v}, \bar{v}]$.*

Proof First we shall define the number $\bar{v}$ by the requirements

$$\bar{v} \geq 2\frac{f(Mh)}{g(Mh)}, \qquad \bar{v} \geq 2\bar{e},$$
$$\frac{h}{2}e^{-h(r+\delta)}(f(\hat{t}) + 0.5\bar{v}g(Mh)) \geq \max_{s\in[0,\omega]} C_i'\left(s, \frac{16e^{\delta h}\bar{e}}{h(\omega - h)g(Mh)}\right). \tag{24}$$

The above definition of $\bar{v}$ applies to the more interesting case $h < \omega$. The alternative case requires minor changes in the last inequality.

After having $\bar{v}$ fixed so that (24) are satisfied we denote

$$\mathcal{V}_h^M(\bar{v}) = \{v = (v_{\hat{k}+1}, \dots, v_M) : |v_k| \leq \bar{v}\}.$$

The proof utilizes the Brouwer fixed point theorem. Therefore we reformulate system (23) as a fixed point equation:

$$F_k(v) = v_k, \quad k = \hat{k}+1, \dots, M, \quad \text{where } F_k(v) = v_k + \beta_k(E_k^{*h}[v] - \hat{E}_k),$$

and $\beta_k > 0$ are chosen in such a way that

$$\beta_k \max_{v\in\mathcal{V}_h^M(\bar{v})} E_k^{*h}[v] \leq \frac{\bar{v}}{2}, \quad \beta_k \leq 1.$$

The above maximum exists due to the continuity of $E_k^{*h}[v]$ and the compactness of $\mathcal{V}_h^M(\bar{v})$.

We shall prove that $F(v) = (F_{\hat{k}+1}(v), \dots, F_M(v))) \in \mathcal{V}_h^M(\bar{v})$ for $v \in \mathcal{V}_h^M(\bar{v})$, which would imply the claim of the proposition due to the Brouwer fix point theorem, since F is continuous and $\mathcal{V}_h^M(\bar{v}) \subset \mathbf{R}^{M-\hat{k}}$ is convex and compact.

Take an arbitrary $v \in \mathcal{V}_h^M(\bar{v})$ and $k \in \{\hat{k}+1, \dots, M\}$. All we have to prove is that $-\bar{v} \leq F_k(v) \leq \bar{v}$.

To prove the second inequality we consider two cases:

1. Let $v_k \geq \bar{v}/2$. Then for $\tau \in [0, Mh]$ we have $f(\tau) - v_k g(\tau) \leq f(Mh) - 0.5\bar{v}g(Mh) \leq 0$ according the first inequality in (24). Hence (see (9)) $E_k^{*h}[v] = 0$ and $F_k(v) = v_k - \beta_k\hat{E}_k \leq v_k \leq \bar{v}$.

2. Let $v_k < \bar{v}/2$. Then $F_k(v) \leq v_k + \beta_k E_k^{*h}[v] \leq \bar{v}/2 + \bar{v}/2 = \bar{v}$ according to the choice of β_k.

To prove that $F_k(v) \geq -\bar{v}$ we consider the next two cases.

3. Let $v_k \geq -\bar{v}/2$. Then $F_k(v) \geq -\bar{v}/2 - \beta_k\hat{E}_k \geq -\bar{v}/2 - \bar{e} \geq -\bar{v}$ according to the second inequality in (24).

4. Let $v_k < -\bar{v}/2$. Clearly $W^*[v](\tau, s) = 1$ if $\tau + s \in [t_{k-1}, t_k]$. Then from the adjoint equation (8) we have that for $\tau \in [t_k - \omega, t_{k-1}]$

$$\xi[v](\tau, s) = \int_s^\omega e^{-(r+\delta)(\theta - s)}(f(\tau) - v(\tau+\theta)g(\tau))W^*[v](\tau, \theta)d\theta$$

and if $s \in [t_{k-1} - \tau, t_k - \tau]$ then

$$\xi[v](\tau, s) \geq \int_s^{t_k - \tau} e^{-(r+\delta)(\theta - s)}(f(\tau) - v_k g(\tau))d\theta,$$

and if $s \in [t_{k-1} - \tau, t_{k-1} - \tau + h/2]$ then

$$\xi[v](\tau, s) \geq e^{-(r+\delta)h}\frac{h}{2}\left(f(\hat{t}) + \frac{\bar{v}}{2}g(Mh)\right).$$

Thus for $\tau \in [t_k - \omega, t_{k-1}]$ and $s \in [t_{k-1} - \tau, t_{k-1} - \tau + h/2]$

$$I^*[v](\tau, s) \geq (C_i')^{-1}\left(s, e^{-(r+\delta)h}\frac{h}{2}\left(f(\hat{t}) + \frac{\bar{v}}{2}g(Mh)\right)\right) \geq \frac{16e^{\delta h}\bar{e}}{h(\omega - h)g(Mh)} =: I^{\#},$$

where we use the last inequality in (24). Then from (1) we obtain that for $\tau \in [t_k - \omega, t_{k-1}]$ and $s \in [t_{k-1} - \tau + h/4, t_{k-1} - \tau + h/2]$

$$K^*[v](\tau, s) \geq \frac{h}{4}e^{-\delta h}I^{\#} = \frac{4\bar{e}}{(\omega - h)g(Mh)}.$$

Finally we estimate

$$\begin{aligned}
E_k^{*h}[v] &= \frac{1}{h}\int_{t_{k-1}}^{t_k}\int_{t-\omega}^{t} g(\tau)K^*[v](\tau, t-\tau)W^*[v](\tau, t-\tau)d\tau\, dt \\
&\geq \frac{1}{h}\int_{t_{k-1}+h/4}^{t_{k-1}+h/2}\int_{t_k-\omega}^{t_{k-1}} g(\tau)K^*[v](\tau, t-\tau)W^*[v](\tau, t-\tau)d\tau\, dt \\
&= \frac{1}{h}\int_{t_k-\omega}^{t_{k-1}}\int_{t_{k-1}-\tau+h/4}^{t_{k-1}-\tau+h/2} g(\tau)K^*[v](\tau, s)ds\, d\tau \\
&\geq \frac{1}{h}(\omega - h)\frac{h}{4}g(Mh)\frac{4\bar{e}}{(\omega - h)g(Mh)} = \bar{e}.
\end{aligned}$$

Using this we obtain $F_k(v) \geq v_k + \beta_k(E_k^{*h}[v] - \hat{E}_k) \geq v_k \geq -\bar{v}$. This proves the invariance of $\mathcal{V}_h^M(\bar{v})$ with respect to F and the proposition. □

Thus Proposition 3 ensures at least a positive answer to the issue (i) at the beginning of the section at least for the truncated equation (23). Then in the special case where assumption (A4) holds and (23) has a regular solution v with the natural choice of the parameters $v_{M+1} = \cdots = v_{M+m} = 0$, this must be the unique regular solution (due to the freedom in the choice of the right-hand side $\hat{E}_k$). Issue (iii) from the beginning of this section remains unclear, and will be somewhat enlightened by the numerical experiments in the next section. Here we only mention that under (A4) and the assumption $\hat{E}(\hat{t}) = E^*[0](\hat{t})$ (that is if the cap starts from the unconstrained emission at $\hat{t}$) the market equation (18) cannot have a regular positive solution. Indeed, if it has a regular solution v, then v solves also (19). This equation applied for $t = \hat{t}$ shows that v cannot be positive on $(\hat{t}, \hat{t} + \omega)$ due to Lemma 4.

5 Numerical Analysis of the Market Price of Emission

In this section we present some experimental results for the auction price of permits assuming that the firms participating to the auction are identical (thus the price is determined by (18), if a positive solution exists). In particular we investigate the role of the aggregation step h. Following the terminology used under the Kyoto protocol, we shall call it a *commitment period*, and h will be its length.[4] The existence and the regularity of the auction price for emission permits will be scrutinized for different h.

The following data specifications are used in the experiments:

$T = 120$—time horizon;
$\omega = 20$—maximal age of capital;
$\delta = 0.1$—depreciation rate of each technology;
$r = 0.04$—discount rate;
$K_0(\tau)$ for vintages $\tau < 0$ is obtained by solving the firm's optimization problem with $v(\cdot) \equiv 0$ on $[-\omega, T]$;
$K_0(\tau) = 0$ for vintages $\tau > 0$;
$f(\tau) = 3 + 0.03\tau$ for $\tau \in [-\omega, T]$—the productivity of one unit of capital of vintage τ;
$g(\tau) = \frac{2.5}{1.7+ln(2+\omega+\tau)} + 0.0002$ for $\tau \in [-\omega, T]$—the emissions produced by one unit of capital of vintage τ;
$C(s, i) = 20(1 - s/\omega)i + 0.5i^2$—the cost of investment i in s-years-old machines of any vintage;

In the figures below the plotted time horizon is 80 years. However, the calculations are made for a time horizon of 120 years, in order to eliminate the truncation error. The results obtained with larger time horizons are visually indistinguishable from the ones given below.

In the first group of plots (Figs. 1 and 2) a constant emission cap $\hat{E}(t) = 300$ is imposed at time $\hat{t} = 30$. This cap is below the level of emission that would be produced without any emission restriction (represented by the dash-dotted line in Fig. 1). The solid line represents the emission of the firm obeying the cap. It is remarkable that the reduction of the emission of the firm begins much earlier than $\hat{t} = 30$—this is the so called anticipation effect. The left plot in Fig. 2 represents the price of the emission permits in the primary market (as auctioned). In fact, it is obtained by using a small commitment periods of $h = 1$ year. The price is highly oscillating close to the time at which the cap is imposed, as it is expected from the explanations given at the beginning of Sect. 4. The right plot corresponds to a commitment periods of $h = 5$ years, for which the fluctuations of the price are substantially lower. Although the primary market is efficient in the considered test example (that is, it determines a positive price of the emissions) a larger commitment period h clearly decreases the volatility of the market.

[4]Under the Kyoto protocol of the UN Framework Convention on Climate Change, the (first) commitment period is a five-year period covering 2008 to 2012.

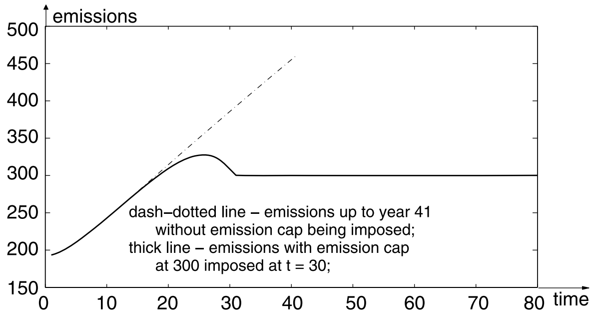

Fig. 1 Emission resulting from an emission restriction posed at time $t = 30$

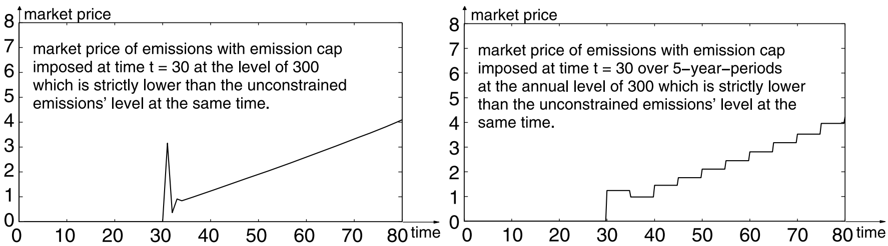

Fig. 2 Auction price of permits with commitment periods of 1 year (*left*) and auction price with commitment period of 5 years (*right*) resulting from an emission restriction posed at time $t = 30$

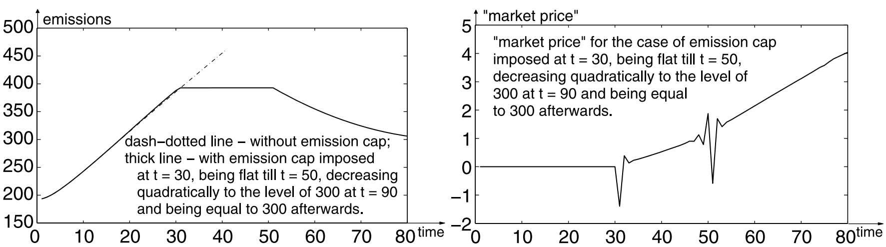

Fig. 3 Emissions (*left*) and a solution to the price equation (18) (*right*) resulting from an emission restriction posed at time $t = 30$

In Fig. 3 the emission cap imposed at time $t = 30$ is first constant, starting from the emission level before the beginning of the emission restriction (the dash-dotted line represents the unrestricted emissions), then it decreases quadratically. The solid line represents the emission of the firm obeying the cap. Here it is remarkable that (in contrast to the first scenario) there is no visible reduction of the emission level before time $t = 30$, i.e. no visible anticipation effect. This phenomenon will be discussed in more detail in the continuation of the paper.

The right plot in Fig. 3 represents the auctioned price of the emission permit. As before it is obtained by using small commitment periods of $h = 1$ year. In contrast to the first scenario, here the auction fails in two time-periods. First this happens immediately after the introduction of the emission cap at time $\hat{t} = 30$, then around

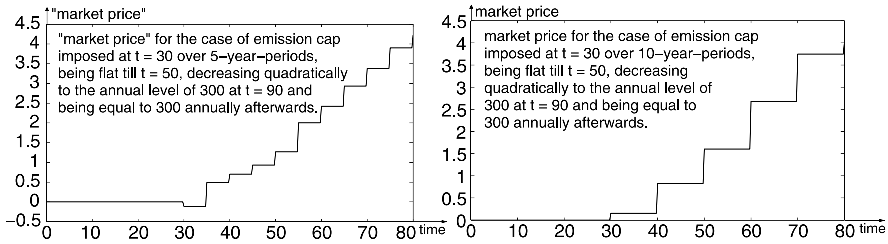

Fig. 4 Auction price of emission permits for commitment periods $h = 5$ years (*left*) and $h = 10$ years (*right*)

the change of the shape of the cap in year $t = 50$: the solution of the auction equation (18) takes negative values. In addition, the solution is highly oscillating around these times. The regularization of the market by posing the cap in commitment periods is seen on Fig. 4. The left plot corresponds to commitment periods of $h = 5$ years. The market still fails immediately after the introduction of the cap, but not around the change of the shape of the cap. The right plot in Fig. 4 represents the price function with commitment periods of 10 years. The market is now efficient and looks quite regular.

References

Barucci, E., & Gozzi, F. (1998). Investment in a vintage capital model. *Research in Economics*, *52*, 159–188.

Barucci, E., & Gozzi, F. (2001). Technology adoption and accumulation in a vintage capital model. *Journal of Economics*, *74*, 1–38.

Boucekkine, R., de la Croix, D., & Licandro, O. (2004). Modelling vintage structures with DDEs: principles and applications. *Mathematical Population Studies*, *11*, 11–29.

Boucekkine, R., Licandro, O., Puch, L., & del Rio, F. (2005). Vintage capital and the dynamics of the AK model. *Journal of Economic Theory*, *120*, 39–72.

Delves, L. M., & Mohamed, J. L. (1985). *Computational methods for integral equations*. Cambridge: Cambridge University Press.

Feichtinger, G., Hartl, R. F., Kort, P. M., & Veliov, V. M. (2006). Anticipation effects of technological progress on capital accumulation: a vintage capital approach. *Journal Economic Theory*, *126*, 143–164.

Feichtinger, G., Hartl, R. F., Kort, P. M., & Veliov, V. M. (2008). Financially constrained capital investments: the effects of disembodied and embodied technological progress. *Journal of Mathematical Economics*, *44*, 459–483.

Finkelshtain, I., & Kislev, Y. (1997). Prices versus quantities: the political perspective. *Journal of Political Economy*, *105*(1), 83–100.

Hansen, P. C. (1992). Numerical methods for analysis and solution of Fredholm integral equations of the first kind. *Inverse Problems*, *8*, 849–872.

Kelly, D. L. (2005). Price and quantity regulation in general equilibrium. *Journal of Economic Theory*, *125*, 36–60.

Kress, R. (1989). *Applied Mathematical Sciences: Vol. 82*. *Linear integral equations*. Berlin: Springer.

Krysiac, F. C. (2008). Prices vs. quantities: the effects on technology choice. *Journal of Public Economics*, *92*, 1275–1287.

Pontryagin et al. (1962). *The mathematical theory of optimal processes*. New York: Wiley.
Tikhonov, A. N., & Arsenin, V. Y. (1977). *Solutions of Ill-posed problems*. New York: Wiley.
Veliov, V. M. (2008). Optimal control of heterogeneous systems: basic theory. *Journal of Mathematical Analysis and Applications*, *346*, 227–242.
Weitzman, M. L. (1974). Prices versus quantities. *Review of Economic Studies*, *41*, 477–491.
Zhao, J. (2003). Irreversible abatement investment under cost uncertainties: tradable emission permits and emission charges. *Journal of Public Economics*, *87*, 2765–2789.