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Fairness and well-being measurement*

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Abstract

We assume that economic justice requires resources to be allocated fairly, and we construct individual well-being measures that embody fairness principles in interpersonal comparisons. These measures are required to respect agents' preferences. Across preferences well-being comparisons are required to depend on comparisons of the bundles of resources consumed by agents. We axiomatically justify two main families of well-being measures reminiscent to the ray utility and money-metric utility functions.

JEL Classification: D63, I32.

Keywords: fairness, well-being measure, preferences.

1 Introduction

Economists evaluate social and economic policies based on their impact on agents' well-being. Given that policies that benefit all agents are unfrequent, that requires some comparability across agents' well-being. One main theory

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of well-being used by economists consists of comparing agents' well-being on the ground of the bundles of resources that they consume. There are cases in which this is easily done. If all agents are assumed to have the same preferences, as it is the case in the optimal taxation literature following Mirrlees' (1971) seminal contribution, then the well-being measure is simply required to be consistent with these common preferences. If all agents have (possibly heterogenous) quasi-linear preferences in money, then the money measure of satisfaction level is natural and creates an easy way of comparing well-being. Identical or quasi-linear preferences are extremely common assumptions.

It is not always relevant to make those assumptions, however. If some agents are close to their liquidity constraints, for instance, it is hard to assume away all income effects. One may also wish to take account of agents' different ways of reacting to policies. As soon as one acknowledges that there are income effects and that agents have heterogenous preferences, it is no longer clear how well-being should be measured.

Many authors have directly or indirectly studied the construction of well-being measures. This construction has been studied directly in the literature on consumer surplus. This abundant literature has culminated in Samuelson's (1974) and Samuelson and Swamy's (1974) concept of money-metric utility, and Samuelson's (1977) and Pazner's (1979) concept of ray utility, that will play a crucial role in what follows. The money-metric utility consists in a priori fixing a vector of prices and measuring well-being by the budget, at those prices, that leaves the agent indifferent with her actual consumption. The ray utility consists in a priori fixing a ray of goods in the consumption set of the agents and measuring well-being by the only bundle of resources along that ray that leaves an agent indifferent with her actual consumption.

The question of how to measure well-being has also been raised indirectly in the literature on fair allocation. In this literature, economic justice is conceived as equality in the way resources are allocated among agents. The formal study of economic justice as fairness began with Kolm (1968, 1972) and Varian's (1974) works on no-envy and Pareto efficiency in private good models. Later on, studies have been extended so as to include a large variety of fairness properties (see, among many others, Moulin, 1996, for a general discussion of lower and upper bounds on welfare) in a large variety of models (see, among many others, Moulin, 1987, for a study of fairness with public goods). Hervé Moulin has been one of the main leaders in these developments (see, for instance, the survey Moulin and Thomson, 1997). Solutions from

that literature can be seen as answering simultaneously the following two questions: how to measure individual well-being and how to aggregate it over the population (see the recent surveys in Thomson, 2011, and Fleurbaey and Maniquet, 2011).

In this paper, we stick to the idea that well-being should reflect the individual value of bundles of resources, but we disentangle the question of how to measure individual well-being from the question of how to aggregate it, and we focus on the former (see Bossert and Weymark, 2004, or d'Aspremont and Gevers, 2002, for detailed surveys of the theory of well-being aggregation). Starting with an abstract model of consumption set, we axiomatically study how to construct well-being measures when well-being is evaluated at a bundle of goods on the basis of the preferences of the agent consuming that bundle. This is in line with undertakings recently launched by Fleurbaey and Tadenuma (2014) and Fleurbaey and Blanchet (2013).

We study the case in which goods are infinitely divisible and more of any good is always preferred to less. Two families of measures emerge. One family is consistent with the idea that comparing well-being requires to determine worst preferences. Worst preferences are preferences that make the experience of consuming any bundle of resources worse than with any other preferences. Worst preferences are naturally connected to the difficulty of trading off between goods. The other family is consistent with the idea that comparing well-being requires to determine best preferences. Best preferences are the ones that make the experience of consuming any bundles of resources better than with any other preferences. Best preferences are naturally connected to the ease with which one is able to trade off between goods.

This first set of results sheds some light on the previous literature on well-being measures. Indeed, the ray utility belongs to the first family of well-being measures we obtain. Our results give an axiomatic characterization of that measure, but it also shows that many other measures can be similarly justified.

Money-metric utility belongs to the second family of well-being measures we obtain. Again, our results can be viewed as providing an axiomatic justification to that measure, but they also show that other measures can receive similar justification.

Our results also shed some light on the theory of fair allocation. In that theory, two prominent allocation rules receive considerable justification. An allocation rule identifies the set of best allocations among the feasible

ones. The first one is the egalitarian equivalent allocation rule, introduced by Pazner and Schmeidler (1978) and later characterized, among others, by Moulin (1987) and Sprumont and Zhou (1999). It consists in allocating goods in such a way that each agent is indifferent between the bundle she is assigned and a common, reference bundle. This is consistent with a way of measuring well-being that belongs to our first family of measures. Of course, the egalitarian equivalent allocation rule also shows how well-being should be aggregated: all agents should have the same well-being.

The second main allocation rule is the equal income Walrasian rule, first studied by Kolm (1968) and Varian (1974). It consists in allocating goods in such a way that the resulting allocation can be thought of a competitive equilibrium allocation from an equal split of the resources. This allocation rule can be decomposed into a way of defining well-being and a way of aggregating it. The way of defining well-being is by looking at equivalence with Walrasian budgets computed at equilibrium prices, those that would prevail if resources were first allocated equally among all agents. This well-being measure belongs to our second family of measures.

The theory of fair allocation has recently looked at social ordering functions instead of allocation rules. A social ordering function is a complete ordering on allocations. The study of social ordering functions, studied in Fleurbaey and Maniquet (2011), has provided us with two main conclusions. The first conclusion is that there was one and only one prominent aggregator of individual well-being levels, the maximin aggregator. That is, simple and weak requirements on social ordering functions force us to maximize the lowest well-being level among agents. The second conclusion is that many different individual well-being measures receive justification from fairness requirements. Many of those measures are of the equivalence type: the well-being of an agent is measured with respect to the bundle of goods, in a set of reference bundles, that leaves this agent indifferent with her actual consumption. Some other measures are closer to the money-metric type: the well-being of an agent is measured with respect to the income that leaves her indifferent to her actual bundle, with prices being chosen so as to maximize the minimal income (see Fleurbaey and Maniquet, 2008 and 2011).

Our results come close to and are inspired by the recent study of Decancq, Fleurbaey and Maniquet (2014) on poverty measures. A measure of individual poverty is no more than the inverse of a well-being measure. The authors of that paper axiomatized a poverty measure that consists in first defining an individual poverty measure consistent with the ray utility func-

tion and then aggregating individual poverty in a way that is only required to be consistent with dominance. Some of their results are reproduced here in the limited frame of well-being measurement.

The well-being measures that we justify in this paper are consistent with the view that economic justice arises from a fair allocation of resources. The measures we propose are solutions to the difficulty arising from the heterogeneity of preferences. We have not addressed, however, the difficulty arising from heterogeneity in needs or in abilities.

The remainder of the paper is organized as follows. In Section 2, we present the model, we define what a well-being measure is, and we state the basic property such a measure needs to satisfy. In Section 3, we strengthen the basic property in one direction, and we prove that the new property leads us to a family of well-being measures that includes the ray utility. In Section 4, we strengthen the basic property in another direction, and we prove that the new property leads us to a family of well-being measures that includes the money-metric. In Section 5, we show how the results of the previous sections are related to the lattice structure of the space of indifference sets. In Section 6, we gather all the proofs. In Section 7, we give some concluding comments.

2 A model of well-being measurement

We assume that there are K divisible goods, and quantities of goods are cardinally measurable (so that, for instance, arithmetic averages of quantities are meaningful). The consumption set is $X = \mathbb{R}^K_+$. Agents have continuous, convex and monotonic[1] preferences over X. We let $\mathcal{R}$ denote the set of all such preferences. A well-being measure is a function $W : X \times \mathcal{R} \to \mathbb{R}$, such that $W(x, R)$ is the well-being level of an agent consuming bundle x and having preferences R. Note that we require from W that it gives us a well-being function for all preferences. This corresponds to the typical universal domain requirement.

Throughout the paper, we require the following two conditions on W. First, W is continuous in x. Second, W respects the preferences, in the sense

[1] We use $>, \geqslant$ and $\gg$ to denote the vector inequalities. Preferences R are monotonic if and only if $x > x'$ implies $x\,R\,x'$ and $x \gg x'$ implies $x\,P\,x'$.

that for all $x, x' \in X$, $R \in \mathcal{R}$,

$$x\, R\, x' \Rightarrow W(x, R) \geq W(x', R) \text{ and } x\, P\, x' \Rightarrow W(x, R) > W(x', R).$$

The latter condition is reminiscent of Pareto efficiency in the social choice literature. Here, it represents our desire to define well-being in a way that is consistent with what agents themselves think about how the different dimensions of life should be aggregated.

The following terminology will prove useful. For $x \in \mathbb{R}^K_+$, $R \in \mathcal{R}$, $L(x, R), U(x, R)$ and $I(x, R)$ denote the lower, upper and indifference (closed) contour of R at x, respectively, that is,

$$\begin{aligned} L(x, R) &= \{x' \in \mathbb{R}^K_+ | x\, R\, x'\}, \\ U(x, R) &= \{x' \in \mathbb{R}^K_+ | x'\, R\, x\}, \\ I(x, R) &= L(x, R) \cap U(x, R). \end{aligned}$$

The first axiom we define captures the idea that well-being comparisons should be made on the basis of the resources that agents consume. We find it a weak axiom and we will impose it throughout the paper. It requires that the well-being of an agent at some bundle be declared larger than that of another agent at another bundle when the following condition is satisfied. Both agents prefer any bundle that the former agent finds indifferent to the former bundle over any bundle that the latter agent finds indifferent to the latter bundle. This condition amounts to assuming that the upper contour set at the former bundle does not intersect the lower contour set at the latter bundle.

Axiom 1 Nested Contour
For all $x, x' \in X$, $R, R' \in \mathcal{R}$, if $U(x, R) \cap L(x', R') = \emptyset$, then $W(x, R) > W(x', R')$.

A similar axiom was introduced in Decancq, Fleurbaey and Maniquet (2013) in their study of poverty measurement (individual poverty measures are opposite functions to well-being measures).

We may also note a relationship between this axiom and the celebrated no-envy condition of the literature on fair allocation. An allocation is a no-envy allocation if no agent strictly prefers the bundle assigned to another agent to her own. It is known that an envy-free allocation may be Pareto indifferent to another allocation in which one agent envies another agent. In

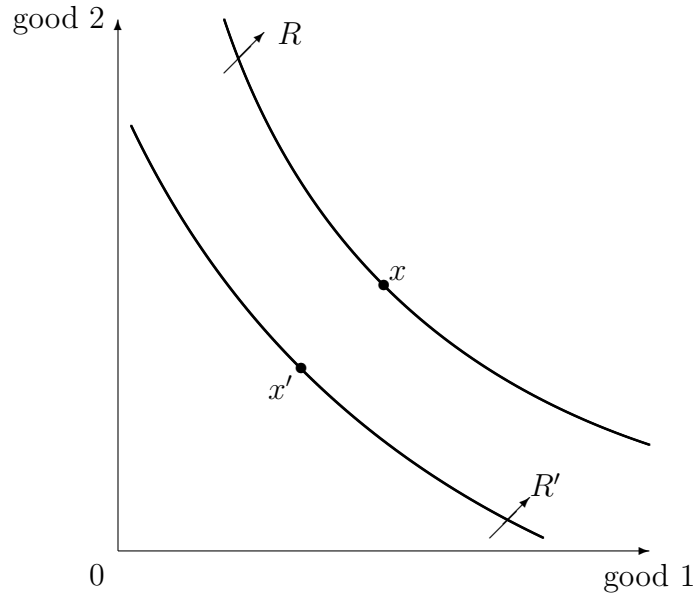

Figure 1: *Nested Contour*: $W(x, R) > W(x', R')$.

terms of well-being measurement, that means that we cannot base our well-being comparisons on envy considerations. A well-being measure that would stipulate that an agent envying another agent's consumption has a lower well-being is impossible to define. Let us think, indeed, of a pair of bundles $x, x' \in X$ and a pair of preferences $R, R' \in \mathcal{R}$ with crossing indifference sets such that $x' \, P \, x$ and $x \, P' \, x'$. It would be impossible to give values to $W(x, R)$ and $W(x', R')$, as the no-envy property would read $W(x, R) < W(x', R')$ and $W(x', R') < W(x, R)$.

The above axiom offers a way to reconcile no-envy and the requirement that a well-being measure be respectful of individual preferences. In the comparison considered in the axiom, agent R', who consumes x', envies agent R, who consumes x, but, moreover, agent R' consuming any bundle she deems equivalent to x' would envy agent R consuming any bundle she deems equivalent to x.

Nested Contour is equivalent to the following axiom. It requires that the well-being of an agent consuming a given bundle only depend on her indifference set at that bundle.

Axiom 2 Unchanged Indifference Independence
For all $x \in X$, $R, R' \in \mathcal{R}$, if $I(x, R) = I(x, R')$, then $W(x, R) = W(x, R')$.

Lemma 1 *A well-being measure W satisfies* Nested Contour *if and only if it satisfies* Unchanged Indifference Independence.

Unchanged Indifference Independence is also reminiscent of a series of axioms in the theory of fair allocation and social ordering functions that require independence of the selection of the best allocation to changes in preferences that do not affect the indifference set through the selection.

We consider that a well-being measure violating *Nested Contour* would be hard to justify, if one wants to be consistent with the idea that economic justice comes from equality of resources. Our strategy from now on will be to propose strengthenings of *Nested Contour* and to study their implications. We will propose two ways of strengthening the axiom, which will turn out to be incompatible with each other, and that will lead us to define two families of well-being measures.

3 Supremum Nested Contour

Nested Contour may be redefined in the following way. If the lower contour set of one agent at her consumption lies in the interior of the lower contour set of another agent at her consumption, then the well being of the former agent is strictly lower than that of the latter agent. This idea can be immediately extended to include several agents in the following way. If the lower contour set of one agent lies in the interior of the union of a countable number of lower contour sets of other agents, then the well being of the former agent is strictly lower than that of at least one of the latter agents.

Axiom 3 Supremum Nested Contour
For all $x \in X$, $R \in \mathcal{R}$, and a countable set $\mathcal{X} \subset X \times \mathcal{R}$, if $L(x, R) \subset$ interior$[\cup_{(x',R') \in \mathcal{X}} L(x', R')]$, then $W(x, R) < \sup_{(x',R') \in \mathcal{X}} \{W(x', R')\}$.

This axiom is clearly more controversial than *Nested Contour*. It is illustrated in Fig. 2 in the case in which $\mathcal{X}$ contains two elements, (x', R') and (x'', R''). The situation is that bundle x consumed by agent R is either considered worse than x' by agent R', or worse than x'' by agent R'', or both. This is why it would be hard to justify that agent R is strictly better-off than both other agents.

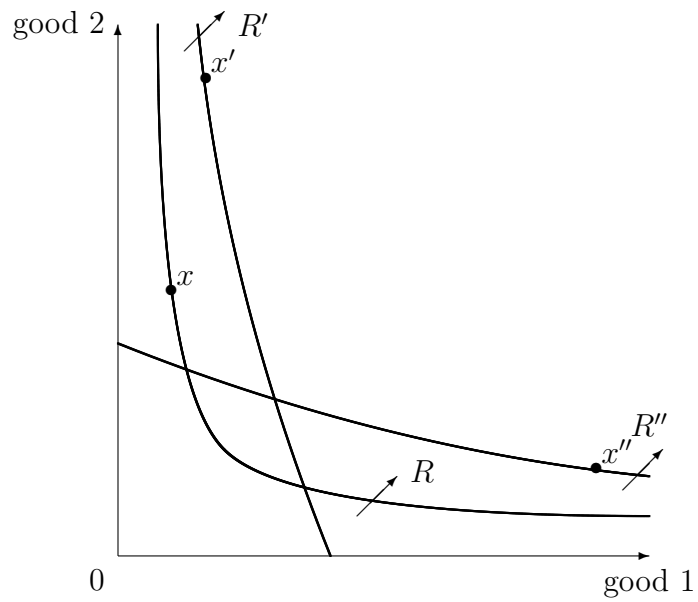

Figure 2: *Supremum Nested Contour*: $W(x, R) < \max\{W(x', R'), W(x'', R'')\}$.

Our first result provides us with a characterization of the well-being measures that satisfy *Supremum Nested Contour*. This characterization is in terms of the existence of worst preferences. We can say that preferences $R^w \in \mathcal{R}$ are the worst preferences if the well-being of an agent with those preferences is always lower than that of any other agent whatever the (common) bundle they consume. As stated in the theorem below, not all preferences $R \in \mathcal{R}$ can play the role of worst preferences. Worst preferences R^w must satisfy the property that they have a maximal element in all lower contour sets and $W(\cdot, R^w)$ increases continuously with these lower contour sets. We let $\mathcal{R}^w$ denote the subdomain of preferences exhibiting this property. We provide a characterization of this subdomain in the appendix.

Axiom 4 Worst Preferences
There exists $R^w \in \mathcal{R}^w$ such that for all $x \in X$, $R \in \mathcal{R}$, $W(x, R^w) \leq W(x, R)$.

Supremum Nested Contour turns out to be equivalent to *Nested Contour* and *Worst Preferences*. Moreover, once worst preferences R^w are chosen and the well-being measure of the worst preferences $W(\cdot, R^w)$ is determined, we have a unique and well-defined well-being measure.

Theorem 1 *A well-being measure W over X satisfies* Supremum Nested Contour *if and only if it satisfies* Nested Contour *and* Worst Preferences. *Moreover, for all $x \in X$ and $R \in \mathcal{R}$:*

$$W(x, R) = \max_{x' \in L(x,R)} W(x', R^w).$$

Worst Preferences involves comparability between any preferences and the worst preferences. The theorem proves that when it is combined with *Nested Contour*, it constructs comparability between any pair of preferences.

At this stage, any preference relation $R \in \mathcal{R}^w$ can be chosen to be the worst one and the axioms are satisfied. Some preferences, though, are more natural candidates to be worst preferences than others. We think that Leontief preferences are natural candidates. Preferences R^ℓ are Leontief if there exist $\ell \in int\Delta^{K-1}$ (where Δ^{K-1} denotes the $K-1$-dimensional simplex), such that

$$x \, R^\ell \, x' \Leftrightarrow \min_{k \in K} \frac{x_k}{\ell_k} \geq \min_{k \in K} \frac{x'_k}{\ell_k}.$$

An agent with Leontief preferences is unable to substitute one good for another. When such an agent consumes bundle x, her well-being is entirely

determined by $\min_{k \in K} \frac{x_k}{\ell_k}$, that is, the good in which this agent feels most deprived.

If a well-being measure satisfies *Worst Preferences* with $R^w = R^\ell$ for some $\ell \in \mathbb{R}^K$, then the well-being of an agent is measured by the bundle that is proportional to ℓ and to which this agent is indifferent. To put it differently, we say that $W(x, R) = W(x', R')$ if and only if there exists some number $\lambda \in \mathbb{R}_+$ such that $x \, I \, \lambda\ell$ and $x' \, I' \, \lambda\ell$ as well. See Fig. 3 for an illustration. All the well-being measures satisfying this property are ordinally equivalent to the ray utility W^ℓ, defined by: for all $x \in X$, all $R \in \mathcal{R}$,

$$W^\ell(x, R) = w \Leftrightarrow x \, I \, w\ell.$$

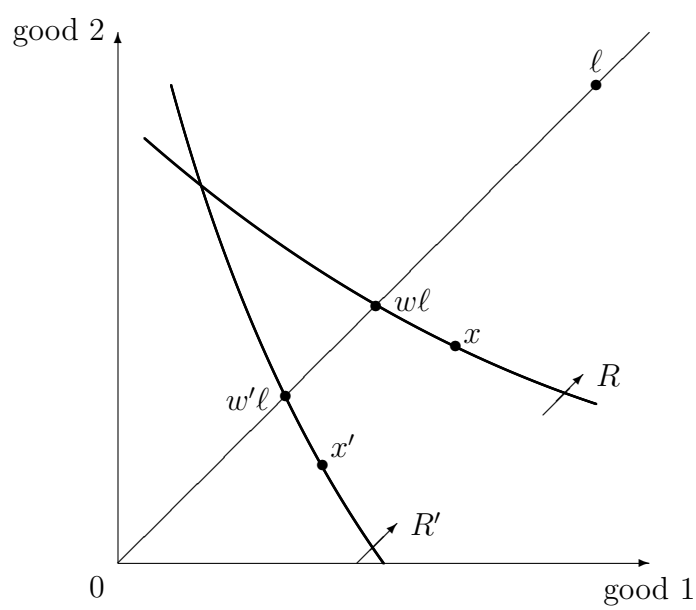

Figure 3: Illustration of W^ℓ: $W^\ell(x, R) = w, W^\ell(x', R') = w'$.

Observe that the set of bundles $r \in \mathbb{R}^K_+$ such that $r = \lambda\ell$ for some $\lambda \in \mathbb{R}_+$ is a ray in $\mathbb{R}^K_+$. This way of measuring well-being was suggested by Samuelson (1977), Pazner (1979) and Deaton (1979). If it is combined with an egalitarian aggregator, it comes close to allocation rules and social ordering functions axiomatized in the literature on fair allocation (see, for instance, Pazner and Schmeidler, 1978, and Fleurbaey, 2005).

4 Infimum Nested Contour

In this section, we study another strengthening of *Nested Contour*. It also refers to the idea that some preferences are intermediary. Consider $x, x', x'' \in X$ and $R, R', R'' \in \mathcal{R}$. One obvious way in which we could say that x is intermediary between x' and x'' is if x lies between the two other bundles, that is $x = \alpha x' + (1-\alpha)x''$, for some $\alpha \in [0,1]$. Of course, this is not sufficient to conclude that we should have $W(x,R) \in [W(x',R'), W(x'',R'')]$ because both x' and x'' might be good for R' and R'' respectively, in some sense of "goodness", whereas x is not that good for R. This can be avoided if we furthermore require that all bundles that are indifferent to x for R be also intermediary between two bundles that are indifferent to x' for R' and x'' for R''. More formally, (x,R) is intermediary between (x',R') and (x'',R'') if for all $y \in X$ such that $y\, I\, x$, there exist $y', y'' \in X$ such that $y'\, I'\, x'$ and $y''\, I''\, x''$ and $y = \alpha y' + (1-\alpha)y''$, for some $\alpha \in [0,1]$. Observe that this is equivalent to requiring that $U(x,R)$ be included in the convex hull of the union of $U(x',R')$ and $U(x'',R'')$.[2] We can then state the axiom that if (x,R) is intermediary between (x',R') and (x'',R'') in this sense, then the well-being at (x,R) must be greater than either that of (x',R'), or that of (x'',R''), or both. Again, we strengthen the requirement so that (x,R) is intermediary between pairs of situations in a countable set of (x',R')'s. Let CH denote the convex hull operator.

Axiom 5 Infimum Nested Contour
For all $x \in X$, $R \in \mathcal{R}$, and a countable set $\mathcal{X} \subset X \times \mathcal{R}$, if $U(x,R) \subset interior[CH(\cup_{(x',R')\in\mathcal{X}} U(x',R'))]$, then $W(x,R) > \inf_{(x',R')\in\mathcal{X}}\{W(x',R')\}$.

Observe that *Infimum Nested Contour* is logically stronger than *Nested Contour* (simply assume that $\mathcal{X}$ is a singleton).

Our next result provides us with a characterization of the well-being measures that satisfy *Infimum Nested Contour*. This characterization is in terms of the existence of best preferences. We can say that preferences $R^b \in \mathcal{R}$ are the best preferences if the well-being of an agent with those preferences is always above that of any other agent whatever the (common) bundle they consume. Again, not all $R \in \mathcal{R}$ can play the role of best preferences. For a similar reason as for worst preferences above, preferences R^b must satisfy the property that it has a minimal element in all upper contour sets, which can

[2] The convex hull of a set is the smallest convex set containing that set.

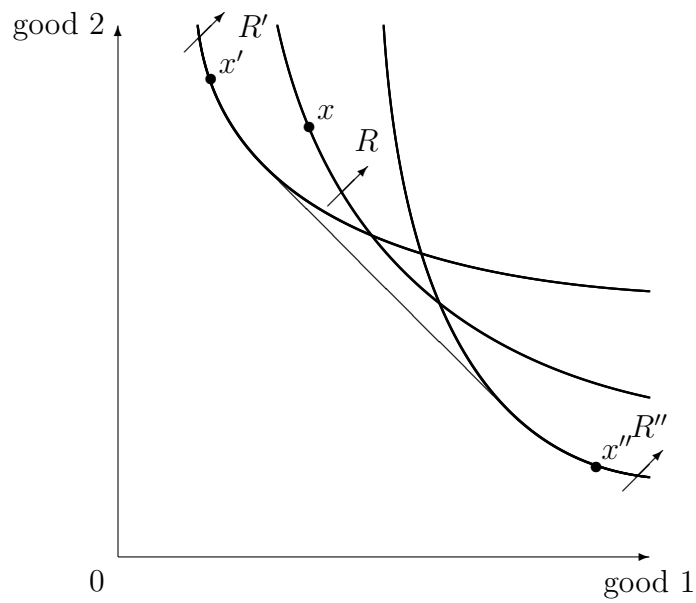

Figure 4: *Infimum Nested Contour*: $W(x, R) > \min\{W(x', R'), W(x'', R'')\}$.

be shown to be equivalent to all $L(x, R^b)$ being compact. We let $\mathcal{R}^b$ denote the subdomain of preferences satisfying this property.

Axiom 6 Best Preferences
There exists $R^b \in \mathcal{R}^b$ such that for all $x \in X$, $R \in \mathcal{R}$, $W(x, R^b) \geq W(x, R)$.

Infimum Nested Contour turns out to be equivalent to *Nested Contour* and *Best Preferences*. Moreover, once best preferences R^b are chosen and the well-being measure of the best preferences $W(\cdot, R^b)$ is determined, we have a unique and well-defined well-being measure.

Theorem 2 *A well-being measure W over X satisfies* Infimum Nested Contour *if and only if it satisfies* Nested Contour *and* Best Preferences. *Moreover, for all $x \in X$ and $R \in \mathcal{R}$:*

$$W(x, R) = \min_{x' \in U(x,R)} W(x', R^b).$$

This theorem offers a nice dual result to the characterization of the well-being measure satisfying *Supremum Nested Contour*. It is interesting and

maybe surprising that *Supremum Nested Contour* and *Infimum Nested Contour* have similar implications. Comparability among preferences is obtained by relationship to some reference preferences. In the former case, the reference preferences always give a lower bound on well-being. In the latter case, they give an upper bound.

Again, *Best Preferences* forces us to choose one preference relation in $\mathcal{R}^b$, but there is no restriction on that choice. It is natural, though, to consider that some preferences are better candidates than others. We think that linear preferences are natural candidates. Preferences R^p are linear if there exist $p \in int\Delta^{K-1}$ such that

$$x\, R^p\, x' \Leftrightarrow \sum_{k \in K} p_k x_k \geq \sum_{k \in K} p_k x'_k.$$

An agent with linear preferences has the highest ability to substitute one good for another. All goods are equally valuable, whatever the proportion in which they come, as soon as we weight them with the p_k parameters.

If we take linear preferences for the best preferences, then the well-being measures characterized in the above theorem are ordinally equivalent to the money-metric utility, introduced by Samuelson (1974) and Samuelson and Swamy (1974). In their definition, the p vector stands for a vector of prices, and the money-metric utility at (x, R) is the minimal expenditure a consumer with preferences R would incur, facing price vector p, to reach the same satisfaction as at x. Instead of relying on the expenditure function terminology, we can define that well-being measure W^p by using the following function: for a set of bundles $B \subset X$, for $R \in \mathcal{R}$, we write $\max(R, B)$ to denote any bundle in B that maximizes R over B, that is, $\max(R, B) = x$ only if $x \in B$ and $x\, R\, x'$ for all $x' \in B$. For all $x \in X$, all $R \in \mathcal{R}$,

$$W^p(x, R) = w \Leftrightarrow x\, I\, \max(R, \{x' \in X | px' \leq w\}).$$

The literature on money-metric utility has failed to provide a convincing way of choosing the reference price vector. Our study does not allow us to make any progress on that issue. One would need more information on the different goods to be able to choose among the possible p's.

The literature on fair allocation has often justified the equal income Walrasian allocation rule as a prominently fair one. The relationship with our result is clear. That rule equalizes the well-being of all agents when one uses

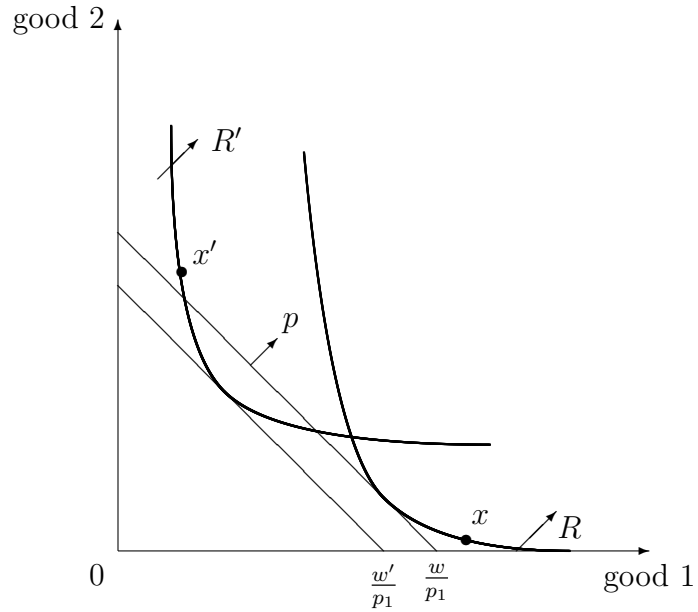

Figure 5: Illustration of W^p: $W^p(x, R) = w, W^p(x', R') = w'$.

the well-being measure characterized by either *Infimum Nested Contour* or *Nested Contour* and *Best Preferences* and the best preferences are linear, with p equal to the equilibrium prices.

5 The space of indifference curves

A consequence of Lemma 1 at the end of Section 2 is that defining well-being measures is equivalent to assigning numbers to indifference curves independently of the precise bundle at which well-being is measured and, more importantly, independently of the preferences to which a given indifference curve belongs. More formally, let $\mathcal{I}$ be the space of all indifference curves, that is, for all $J \in \mathcal{I}$, there exists $R \in \mathcal{R}$ and $x \in X$ such that $J = I(x, R)$. A well-being measure satisfying *Nested Contour* can equally well be defined as a continuous function $W : \mathcal{I} \to \mathbb{R}$ having the property that if J is everywhere above J' then $W(J) > W(J')$.

In this section, we restate the results obtained so far by taking advantage

of the structure of the space of indifference curves.[3]

Abusing on notation, we can adapt our notions of lower and upper contour sets by describing them as:

$$\begin{aligned} L(J) &= L(x,R), x \in J, J = I(x,R), \\ U(J) &= U(x,R), x \in J, J = I(x,R). \end{aligned}$$

We define two relations on $\mathcal{I}$. First, for $J, J' \in \mathcal{I}$, we write

$$J \leq J' \Leftrightarrow L(J) \subseteq L(J').$$

Second, we write

$$J \ll J' \Leftrightarrow L(J) \cap U(J') = \emptyset.$$

The following lemma states the obvious properties of these relations.[4]

Lemma 2 *$(\mathcal{I}, \leq)$ is a lattice, $\ll$ is a strict partial order.*

The supremum and infimum operators of $(\mathcal{I}, \leq)$ are defined as follows:[5] for $J, J', J'' \in \mathcal{I}$,

$$\begin{aligned} J'' = J \vee J' &\Leftrightarrow U(J'') = U(J) \cap U(J'), \\ J'' = J \wedge J' &\Leftrightarrow U(J'') = CH(U(J) \cup U(J')). \end{aligned}$$

The last definition we need is that of a chain. A *chain* C is a set of $J \in \mathcal{I}$ such that

- C covers X: for all $x \in X$, there exists $J \in C$ such that $x \in J$,
- for all $J, J' \in C$, $J \neq J'$, either $J \ll J'$ or $J' \ll J$.

Note that the set of indifference curves of any preference relation $R \in \mathcal{R}$ is a chain.

Equipped with these concepts, we can restate our axioms in the following way.

[3]This section is almost entirely inspired by written comments received from Hervé Moulin on a previous version of the paper.

[4]A lattice consists of a partially ordered set in which every two elements have a unique supremum (also called a join) and a unique infimum (also called a meet).

[5]These definitions come from the following reasoning. The supremum is the minimal element J'' of $\mathcal{I}$ such that $J'' \geq J, J'$, and conversely the infimum is the maximal element J'' of $\mathcal{I}$ such that $J'' \leq J, J'$.

Axiom 7 Nested Contour
For all $J, J' \in \mathcal{I}$, if $J \ll J'$, then $W(J) < W(J')$.

Axiom 8 Supremum Nested Contour
For all $J \in \mathcal{I}$, and $\mathcal{J}$ a countable subset of $\mathcal{I}$, if $J \ll \vee_{J' \in \mathcal{J}} J'$, then $W(J) < \sup_{J' \in \mathcal{J}}\{W(J')\}$.

Axiom 9 Infimum Nested Contour
For all $J \in \mathcal{I}$, and $\mathcal{J}$ a countable subset of $\mathcal{I}$, if $J \gg \wedge_{J' \in \mathcal{J}} J'$, then $W(J) > \min_{J' \in \mathcal{J}}\{W(J')\}$.

The duality between *Supremum* and *Infimum Nested Contour* is even more transparent in this restatement. We can then adapt, for instance, Theorem 1 as follows.

Theorem 3 *If a well-being measure W satisfies* Supremum Nested Contour*, then there exists a chain C such that for all $J \in \mathcal{I}$,*

$$W(J) = \max_{J' \in C : J \not\leq J'} W(J').$$

The dual statement would be a restatement of Theorem 2. The proof of Theorems 1 and 2, given in Section 6 below, only uses the properties of $\leq$ and $\ll$. They could be replaced by a proof only using the terminology of this section.

The lesson to be drawn from this section is that the lattice structure of the space of indifference curves shows a clear duality between *Supremum* and *Infimum Nested Contour*, which transposes into a duality between W^ℓ and W^p, and, by extension, between egalitarian equivalence allocations and equal income allocations. That means that these pairs of solutions are likely to receive dual justifications. That also explains why the literature on well-being measurement, fair allocation or fair social orderings has often converged towards pairs of solutions reminiscent of these two families of solutions.

6 Proofs

Proof of Lemma 1: 1) *Nested Contour $\Rightarrow$ Unchanged Indifference Independence.* Let W satisfy *Nested Contour*. Let $x \in X$ and $R, R' \in \mathcal{R}$ be such that $I(x, R) = I(x, R')$. Assume $W(x, R) \neq W(x, R')$. Let us consider,

without loss of generality, that $W(x,R) > W(x,R')$. By continuity of W, there exists $x' \in X$ such that $x\,P\,x'$ and yet $W(x',R) > W(x,R')$. Note that $I(x,R) = I(x,R')$ implies that $U(x,R') \cap L(x',R) = \emptyset$. By *Nested Contour*, $W(x',R) < W(x,R')$, a contradiction.

2) *Unchanged Indifference Independence* $\Rightarrow$ *Nested Contour*. Let $x, x' \in X$ and $R, R' \in \mathcal{R}$ be such that $U(x,R) \cap L(x',R') = \emptyset$. Let $R'' \in \mathcal{R}$ be such that $I(x,R) = I(x,R'')$ and $I(x',R') = I(x',R'')$. By *Unchanged Indifference Independence*, $W(x,R) = W(x,R'')$ and $W(x',R') = W(x',R'')$. By the assumption that W respects preferences, $W(x,R'') > W(x',R'')$. By transitivity, $W(x,R) > W(x',R')$, which proves the claim. ■

Proof of Theorem 1:1) *Supremum Nested Contour* $\Rightarrow$ *Nested Contour*: It follows from the definition of *Supremum Nested Contour* applied to $\mathcal{X}$ being a singleton.

2) *Supremum Nested Contour* $\Rightarrow$ *Worst Preferences* for some $R^w \in \mathcal{R}^w$. The proof consists of constructing the worst preferences R^w. Let $r \in W(X,\mathcal{R})$. Let $U^r \subseteq X$ be defined as

$$U^r = \bigcap_{(x,R):W(x,R)=r} U(x,R).$$

Let $x^r \in X$, $R^r \in \mathcal{R}$ be such that $U(x^r,R^r) = U^r$. Such R^r exists because U^r is closed and convex and $\mathcal{R}$ contains all continuous, convex and monotone preferences over X. We claim that $W(x^r,R^r) = r$. Assume not. If $W(x^r,R^r) < r$, then there exists $x \in X$ such that $x\,P^r\,x^r$, $W(x,R^r) \leq r$ and $U(x,R^r) \subset interior[U^r] \subset U(x^*,R^*)$ for (x^*,R^*) such that $W(x^*,R^*) = r$. To sum up, we have $W(x,R^r) \leq W(x^*,R^*)$, whereas $U(x,R^r) \cap L(x^*,R^*) = \emptyset$, in contradiction to *Nested Contour*.

If $W(x^r,R^r) > r$, then, by continuity of W, there exists $x \in X$ such that $x^r\,P^r\,x$, $W(x,R^r) > r$ and

$$L(x,R^r) \subset interior\left[\bigcup_{(x,R):W(x,R)=r} L(x,R)\right].$$

That is, $L(x,R^r)$ is covered by the interior of all $L(x,R)$ such that $W(x,R) = r$. As a consequence, it is also covered by a countable subset of those. To put it differently, there exists a countable set $\mathcal{X}$ of (x,R) such that $L(x,R^r) \subset interior[\cup_{(x,R)\in\mathcal{X}} L(x,R)]$. By *Supremum Nested Contour*,

$$W(x,R^r) < \sup_{(x,R)\in\mathcal{X}} \{W(x,R)\} = r,$$

the desired contradiction.

By repeating the argument for all $r \in \mathbb{R}$, we obtain a set of nested convex upper contour sets, which, by continuity of the preferences, form a continuous, convex and monotone preference relation R^w, which, given the universal domain assumption, belongs to $\mathcal{R}$. Note that the construction of R^w, above, implies that for all $x \in X$, $R \in \mathcal{R}$, there exists $x' \in X$ such that $L(x, R) \cap U(x', R^w) = \emptyset$. Indeed, let x'' be such that $x'' \, P \, x$. Let $r = W(x'', R)$. We have $L(x, R) \cap U^r = \emptyset$, since $U^r \subset U(x'', R)$. We may choose x' such that $U(x', R^w) = U^r$.

Before we prove that $R^w \in \mathcal{R}^w$, we need to prove the property of the second statement of the theorem. Let $x \in X$ and $R \in \mathcal{R}$. We claim that

$$W(x, R) = \sup_{x' \in L(x,R)} W(x', R^w).$$

Assume not. First, let us note that $\sup_{x' \in L(x,R)} W(x', R^w)$ exists, because the combination of *Nested Contour* and the property that $L(x, R) \cap U(x', R^w) = \emptyset$ proven above implies that the set of $W(x', R^w)$ for $x' \in L(x, R)$ is bounded above. Assume $W(x, R) < \sup_{x' \in L(x,R)} W(x', R^w)$. Let $x^* \in L(x, R)$ be such that $W(x^*, R^w) > W(x, R)$. We have $W(x^*, R) \leq W(x, R) < W(x^*, R^w)$, a contradiction to *Worst Preferences*.

Assume $W(x, R) > \sup_{x' \in L(x,R)} W(x', R^w) = m$. Note that for $x' \in X$ such that $W(x', R^w) = m$, we have $interior[L(x, R)] \cap U(x', R^w) = \emptyset$. By continuity of W, there exists $x^* \in X$ such that $x \, P \, x^*$, $W(x^*, R) < W(x, R)$, $L(x^*, R) \cap U(x', R^w) = \emptyset$ and yet $W(x^*, R) > m$, in contradiction to *Nested Contour*.

Finally, we claim that

$$\sup_{x' \in L(x,R)} W(x', R^w) = \max_{x' \in L(x,R)} W(x', R^w).$$

Assume not. Let $x^* \in X$ be such that $W(x^*, R^w) = \sup_{x' \in L(x,R)} W(x', R^w)$. The fact that the supremum is not a maximum implies that $L(x, R) \cap U(x^*, R^w) = \emptyset$. By *Nested Contour*, $W(x, R) < W(x^*, R^w)$, a contradiction.

3) *Worst Preferences* for some $R^w \in \mathcal{R}^w$ and *Nested Contour* $\Rightarrow$ *Supremum Nested Contour*: Let $x \in X$, $R \in \mathcal{R}$, and a countable subset $\mathcal{X} \subset X \times \mathcal{R}$ be such that

$$L(x, R) \subset interior[\cup_{(x', R') \in \mathcal{X}} L(x', R')]. \tag{1}$$

We need to show that $W(x,R) < \sup_{(x',R')\in\mathcal{X}} W(x',R')$. By step 3) above,

$$W(x,R) = \sup_{\bar{x}\in L(x,R)} W(\bar{x},R^w)$$

and for all $(x',R') \in \mathcal{X}$,

$$W(x',R') = \sup_{\bar{x}\in L(x',R')} W(\bar{x},R^w).$$

By Eq. 1,

$$\sup_{\bar{x}\in L(x,R)} W(\bar{x},R^w) < \sup_{\bar{x}\in\cup_{(x',R')\in\mathcal{X}} L(x',R')} W(\bar{x},R^w).$$

Besides, we have

$$\sup_{\bar{x}\in\cup_{(x',R')\in\mathcal{X}} L(x',R')} W(\bar{x},R^w) = \sup_{(x',R')\in\mathcal{X}} W(x',R').$$

To sum up, $W(x,R) < \sup_{(x',R')\in\mathcal{X}} W(x',R')$, the desired outcome. ■

Proof of Theorem 2: This proof parallels the proof of Theorem 1. ■

7 Conclusion

Evaluating social policies or assessing the level of social welfare in an economy requires to define ways of aggregating individual measures of well-being. Among the many theories that propose ways of doing it, the theory of fair allocation has proposed solutions based on the idea that economic justice is a matter of fair allocation of resources. The solutions that are proposed in that literature are strongly egalitarian. In the study of social ordering functions, for instance, only maximin types of aggregators turn out to receive axiomatic justification.

Our line of research, in this paper, has been to disentangle the question of measuring individual well-being from the question of aggregating it at the social level. We have focused on the former, and we have developed an axiomatic study of well-being measures. Our main finding is that our axioms turn out to justify two families of well-being measures, each of which contain measures that are common in the whole literature on well-being measurement and fair allocation. The axiomatic justification we give to these two families are dual to each other. One family is consistent with the idea that the

fundamental choice that has to be made is about worst preferences: which preferences make experiencing any bundle the most painful one. It turns out that making such a choice allows one to fully compare well-being between any pair of agents. If worst preferences are of the Leontief type, that is, if they are the preferences of someone who is unable to trade off among goods, then the resulting measure is the ray utility measure of Samuelson (1977), Deaton (1979), Pazner (1979), Pazner and Schmeidler (1978), Decancq, Fleurbaey and Maniquet (2013) and many others.

The other family is consistent with the idea that the fundamental choice is about best preferences: which preferences make experiencing any bundle the least painful one. Again, making such a choice allows one to fully compare well-being between any pair of agents. If best preferences are linear, then the resulting measure is the money-metric utility well-being measure of Samuelson (1974), Samuelson and Swamy (1974) and many others.

This paper presents results about how to measure well-being when the assumption is that economic justice is a matter of fair resource allocation and agents have different preferences. Even if heterogenous preferences raise one of the most challenging questions in well-being measurement and certainly the one that has received the largest attention, it is not the only one. The question of how to measure well-being when agents have heterogenous needs and abilities is certainly worth being addressed as well.

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8 Appendix

In the text, we introduce the condition that $W(\cdot, R^w)$ must "have a maximal bundle over all lower contour sets and increases continuously with these lower contour sets" and the set $\mathcal{R}^w$ of all preferences satisfying this condition. In this appendix we provide a more precise formal definition of this class of preferences.

First let us define the Leontief envelope of a set $A \subset X$ as

$$LE(A) = \partial \bigcap_{x \in X, x \leq A} \{y \in X \mid y \geq x\},$$

where $x \leq A$ means that $x \leq a$ for every $a \in A$, and ∂A denotes the lower frontier of A. It can alternatively be defined as the lower frontier of the smallest set (with respect to set inclusion) that contains A and is a translation of the positive orthant.

The property P that defines the class $\mathcal{R}^w$ is as follows:
(i) for every $x \in X$, there exists a compact $C \subset X$ such that $I(x, R) \setminus C = LE(U(x, R)) \setminus C$;
(ii) for every $x \in X$, there exists $z \in X$ such that $x \leq U(z, R)$.

Condition P(i) says that $I(x, R)$ has the shape of a Leontief indifference set beyond a certain distance from the origin. Condition P(ii) says that $U(z, R)$ goes to infinity in all directions when z goes to infinity.

Proposition 1 *A preference relation $R \in \mathcal{R}$ satisfies property P if and only if for all $R^* \in \mathcal{R}$, $\max\{W(x, R) \mid x \in L(x^*, R^*)\}$ exists for all $x^* \in X$ and is continuous in x^*.*

Proof. Only if: We first prove that $\max\{W(x', R) \mid x' \in L(x, R')\}$ exists. Let $R^* \in \mathcal{R}$, and let $L(x^*, R^*)$ be an arbitrary lower contour set for R^*. By P(ii), there is $z \in X$ such that $x^* \leq U(z, R)$. By monotonicity of preferences, $U(z, R) \subset U(x^*, R^*)$. Consider the collection of sets U such that $U \subset U(x^*, R^*)$ and $U = U(z, R)$ for some $z \in X$. Take its union U^*. By continuity of preferences R and R^*, there is z^* such that $U^* = U(z^*, R)$, and $\inf_{x \in L(x^*, R^*), y \in U^*} \|x - y\| = 0$.

The fact that $\inf_{x \in A, y \in B} \|x - y\| = 0$ can, in general, occur in two ways for closed sets A, B. Either $A \cap B \neq \varnothing$, or $A \cap B = \varnothing$ and there is a sequence $(x_n, y_n) \in A \times B$ such that $\lim \|x_n - y_n\| = 0$. Necessarily this sequence is unbounded (if it were bounded, the sequence would converge since the sets are closed).

Let us apply this to $A = L(x^*, R^*)$, $B = U^*$. The case $A \cap B \neq \varnothing$ provides a maximal bundle, the desired result. The other case cannot occur here because of P(i). Indeed, an unbounded sequence $y_n \in I(z^*, R)$ must end up in the part where $I(z^*, R) \setminus C = LE(U(z^*, R)) \setminus C$. By monotonicity of preferences, a sequence $x_n \in L(x^*, R^*)$ cannot converge toward y_n unless it also belongs to $I(z^*, R) \setminus C$, implying that $A \cap B \neq \varnothing$ again, a contradiction. This completes the proof of existence.

We now prove that $\max\{W(x,R) \mid x \in L(x^*,R^*)\}$ is continuous. First, observe that P(i) implies that for all $x, x' \in X$, $k, \ell \in \{1,\ldots,K\}$, if $x_k = 0 = x'_\ell$, then $x\, I\, x'$ (all bundles with at least one zero coordinate belong to the same indifference set). To see it, apply P(i) to $x = (0,\ldots,0)$. As a result, either $(0,\ldots,0)$ is an argument of $\max\{W(x,R) \mid x \in L(x^*,R^*)\}$ or the argument is an interior point. Therefore, as $L(x^*,R^*)$ grows continuously, so does $\max\{W(x,R) \mid x \in L(x^*,R^*)\}$.

If: We begin by proving that P(i) is necessary. Having a maximal bundle in every lower contour set implies to have a maximal bundle in every lower contour set of Leontief preferences. This clearly implies that $I\,(x,R)$ have the shape of a Leontief indifference set at least beyond a certain distance from the origin. Moreover, continuity of $\max\{W(x,R) \mid x \in L(x^*,R^*)\}$ in x^* implies that the argument of $\max\{W(x,R) \mid x \in L(x^*,R^*)\}$ is either indifferent to $(0,\ldots,0)$ or an interior point. Indeed, assume, on the contrary, that the maximal bundle on some lower contour set is some x' on the boundary of X, with $x'\, P\, (0,\ldots,0)$. Then there exists $x'' = \lambda x' + (1-\lambda)(0,\ldots,0)$, $\lambda \in (0,1)$, such that $x'\, P x''\, P\, (0,\ldots,0)$. Let $R' \in \mathcal{R}$ be such that for all $x \in X$, if $x''\, P\, x$, then $L(x,R') = L(x,R)$ and $L(x'',R') = L(x'',R) \cup \partial X$. We have that for all $x \in X$ such that $x''\, P\, x$, $W(x,R') = W(x,R)$ whereas $W(x'',R') \geq W(x',R)$, a discontinuity.

To see that P(ii) is also necessary, consider the following R, defined over $X = \mathbb{R}^2_+$, satisfying P(i) but not P(ii): R is represented by the following U function: for all $x = (x_1,x_2) \in X$, $U(x) = \min\{\frac{x_1}{1+x_1}, x_2\}$. All indifference sets have the Leontief shape beyond a distance from the origin, but the Leontief parts of these sets cover a set that is bounded in dimension 1. R has no maximal element in, for instance, $L = \{(x_1,x_2) \in X \mid \min\{x_1,x_2\} \leq 2\}$. This case is excluded by P(ii). ■