Cooperation, Competition and Entry in a Tullock Contest

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Cooperation, Competition and Entry in a Tullock Contest

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Abstract

We propose a model of network formation in a Tullock contest. Agents first form their partnerships and then choose their investment in the contest. While a link improves the strength of an agent, it also improves the position of her rival. It is thus not obvious that they decide to cooperate. We characterize all pairwise equilibrium networks and find that the network formation process can act as a barrier to entry to the contest. We then analyze the impact of network formation on total surplus and find that a social planner can increase total surplus by creating more asymmetry between agents, as long as this does not reduce the number of participating agents. We show that barriers to entry may either hurt total surplus, as the winner of the prize does not exploit all the possible network benefits, or improve total surplus since less rent is dissipated when competition becomes less fierce. Finally, when networking acts as an endogenous barrier to entry, no pairwise equilibrium network is efficient.

Keywords: Network Formation, Tullock Contest, Participation Constraints, Efficiency
JEL Classification: D72, D85.

1. Introduction

In many instances in economics and politics competition takes place in contests where competing agents spend resources in order to increase their chance of winning a prize. For instance, firms invest in R&D in order to get a patent and invest in marketing campaigns to increase their market shares, colleagues work hard when they are competing for a promotion, lobbies exert pressure to influence political decisions, etc.[1] Rivals in contests are often involved in bilat-

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    [1]These applications are discussed in detail in the excellent overview on contest theory and its applications by Konrad (2009).

eral cooperative relationships. Arzaghi and Henderson (2008) report that small
advertising agencies share innovative ideas and expertise while competing to
obtain new accounts. Competing firms increasingly share databases concerning
detailed customer information (Liu and Serfers, 2006; Leminen et al., 2008).
Universities cooperate in developing joint IT projects, opening their libraries
to each others' students or sharing software licenses. This paper develops a
two-stage model of network formation among competitors in a Tullock contest
(Tullock, 1980).[2] Agents first form their partnerships and then choose their
investment in the contest. By cooperating, two competitors increase their valu-
ation for the prize, say because they have access to a larger market in case they
get the prize, or because they share information on how to exploit the prize.
Consider cooperation in an R&D contest. Sharing technologies between firms
may decrease the expected time to obtain and market a new product, thereby
increasing the expected value of the "prize". Another example is the sharing of
customer databases between two firms. If they compete for customers through
advertising, it allows firms to have access to a larger market in case they manage
to win an advertising contest, thus increasing the value of the "prize". To fo-
cus on the case that most favors collaboration we assume that linking costs are
negligible.[3] While a link improves the strength of an agent, it also improves the
position of her rival. It is thus not obvious that competitors wish to form links,
even if there is no cost attached to doing so. We show that, even in the case
of negligible costs of link formation, network formation endogenously creates
barriers to entry and deters collaboration. Each network structure defines the
profile of valuations of the competitors and leads to a unique Nash equilibrium
choice of effort in the second stage. Solving the game by backward induction,
we then characterize the set of pairwise equilibrium networks of the link forma-
tion game. This exercise requires to solve an (a)symmetric Tullock contest for
each possible (a)symmetric network configuration and for this reason we limit
ourselves to the case in which valuations depend linearly on the amount of links
an agent has, as we believe this assumption strikes a balance between realism
and tractability.[4] We first show that participating agents are not always better
off by adding a link, i.e. our game does not satisfy link monotonicity (Lemma
1). Nevertheless, we show that two participating agents are always connected in
any pairwise equilibrium (Proposition 1). We then fully characterize all pair-
wise equilibrium networks. They are dominant group networks, i.e. one group of
agents is completely linked to the other agents in the group, and the remaining
agents have no links (Proposition 2). Unconnected agents do not participate to
the contest. Network formation can thus act as a barrier to entry to the con-

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[2]As pointed out in Konrad (2009), the Tullock contest (and its variants) is also used to
study competition for market shares through efforts such as advertising

[3]We assume that benefits of link formation are not (partially) enjoyed by outsiders. Indeed,
assuming that the benefits travel more than distance one would again hamper collaboration.

[4]A possible alternative is to assume that the valuations depend multiplicatively on the
amount of links. All the results we obtain would go through, but we find this assumption less
appealing.

test. We further analyze the impact of network formation on total surplus. We show that a social planner can increase the total surplus by reallocating links so as to increase the asymmetry between agents, as long as it does not reduce the number of participating agents (Proposition 3). Total surplus may decrease, however, if fewer contestants remain active after the social planner implements a spread of links keeping the mean degree of links constant (Proposition 4). Barriers to entry may either hurt total surplus as the winner of the prize does not exploit all the possible network benefits, or improve total surplus since less rent is dissipated as competition becomes less fierce. Finally, we show that when networking acts as a barrier to entry, pairwise equilibrium networks do not maximize total surplus (Proposition 5). From an empirical point of view, network structures of collaboration often display asymmetries, clusters of strongly linked rivals and agents who decide not to cooperate at all (see for instance Hagedoorn and Schakenraad, 1992 and Powel et al., 2005). Hochberg et al. (2010) show that strong network ties among venture capitalists in a given market deters entry. Our theoretical predictions are in line with the observations of clusters and entry deterrence.

Our paper contributes to two strands of literature. First, we contribute to the comparative statics of asymmetric Tullock contests. In a linear Tullock contest, Stein (2002, Proposition 5, p. 331) establishes that the expected payoff of an agent increases with her valuation, provided no agent leaves the contest.[5] We show that it is not necessarily the case when two agents see their valuation increase by the same amount. Stein (2002, Proposition 2, p. 330) also shows that rent dissipation decreases after a mean preserving spread of the agents' valuations, as long as the set of participating agents remains unchanged.[6] We generalize this result by showing that it also holds when the set of participating agents does not decrease after a reallocation of links which increases the asymmetry of the agents' valuations. We then show that Stein's result does not necessarily hold when the set of participating agents decreases after such reallocation as it may increase rent dissipation. Second, our work is closely related to the literature on "co-opetition" in networks. To the best of our knowledge we are the first to present a model of *endogenous network formation* in a Tullock contest with *endogenous efforts* and *endogenous participation constraints*. Many papers analyze horizontal cooperation between oligopolists or between rival firms in R&D races. Goyal and Moraga-Gonzalez (2001) propose a game where firms that compete in the final market form links to cooperate in R&D leading to smaller costs of production. They show by means of an example that asymmetric networks may lead to exclusion and if so, it does not neces-

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[5]Nti (1999, 2004) is among the first to study the comparative statics of asymmetric contests, but restricts attention to the two agents case. Matros (2006) studies the effect of adding or deleting an agent to the contest.

[6]Cornes and Hartley (2005) obtain the same result in contests with convex and non-convex contest success functions. Ryvkin (2013) shows that rent dissipation decreases with a mean preserving increase in the variation of agents' abilities if effort costs are convex and sufficiently steep.

sarily harm total surplus. Goyal and Joshi (2003)[7] develop a model where each partnership translates exogenously and linearly into a marginal cost reduction. Assuming that the parameters are such that all firms are active in the market, they show that the only pairwise stable network is the complete network when the costs of link formation are small. When the costs of link formation are large, they show that any pairwise stable network has a dominant group structure.[8] We obtain instead, that, when linking costs are low, participation constraints are key to the dominant group result. This aspect is absent in Goyal and Joshi's model. Goyal and Joshi (2006) introduce a patent race model in networks, where the period at which a firm expects to innovate is positively correlated to its number of links. They show that the only stable network architecture is the complete network when the cost of links formation is small. Marinucci and Vergote (2011) show that dominant group networks where unconnected agents are left out of the competition are the only pairwise stable networks in an all pay auction in which link formation affects the value of the prize in a multiplicative way. Besides the different set-up, the results of Marinucci and Vergote (2011) depend on a strong assumption on the payoffs: links affect the value for the prize in a multiplicative way. This implies that values increase exponentially with the amount of links one has. We assume instead that values are linear in the amount of links. In addition, they do not study the efficiency properties of the stable networks, while this is an important focus of the current paper. Westbrock (2010) studies efficiency in the model of Goyal and Joshi (2003). He finds that strongly efficient networks must either be dominant group or have the interlinked star architecture. In our model, we show that dominant group networks never maximize total surplus, but the smallest dominant group networks may lead to higher surplus than other pairwise equilibrium networks, including the complete network.[9] Other papers have introduced contests in networks.

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[7]As in Goyal and Joshi (2003), we consider the case where the identity of a partner does not matter. The valuation of an agent only depends on the number of agents with whom she cooperates. In particular, the valuation of an agent is not affected by her partners' efforts. More generally one could introduce spillovers through link formation. Goyal and Moraga-Gonzalez (2001) have introduced R&D spillovers in an oligopoly. Due to the complexity of the model, they derive results only for symmetric networks or in an example among three agents. In the setting of our paper, such modeling strategy would not allow us to analytically solve our two-stage game. We therefore abstract from such spillover effects.

[8]In Goyal and Joshi (2003), the benefits of link formation are increasing in an agent's degree. Thus, there are costs of link formation such that connected agents in a dominant group network prefer to keep their connections while an unconnected agent does not gain sufficiently by forming a new link. Contrary to Goyal and Joshi (2003), the entire distribution of other agents' degrees, and not only their sum, matters in our model. For this reason, we are not able to characterize all pairwise stable networks when linking costs are large. When linking cost are small, dominant group networks are pairwise stable because isolated agents would not benefit from the addition of a new link as they would not participate anyway to the contest. If we only consider parameters values such that each agent participates to the contest, the only pairwise stable network would be the complete network as in Goyal and Joshi (2003). Note that in our paper we characterize pairwise equilibria which are a fortiori pairwise stable.

[9]We thereby complement the results of Belhadj et al. (2015), Billand et al. (2014), Hure-

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Jost (2007) studies a Tullock patent contest where firms choose their R&D investment to win the competition. When firms form a link they share their R&D capacities, leading to free-riding and underinvestment in R&D. In equilibrium all agents are active and have exactly one link. Hiller (2012) proposes a model of signed network formation, where agents extend positive links to others in order to extract rents from enemies. He finds that the network structure that emerges features asymmetric groups of agents, with members of a larger group extracting rents from those in a smaller group. Franke and Ozturk (2009) propose a model where the network exogenously determines the conflictive relations among agents, and relate the network structure to the conflict intensity. Note that in some contests like sports competitions, the contest designer may want to induce contestants to produce the highest possible effort. In such a framework, Franke et al. (2013) study how a central planner should intervene by favoring or handicapping some agents when they have asymmetric abilities. Instead we consider the case in which efforts are considered waste, and focus on whether a mean preserving spread of valuations, through the reallocation of links, increases or decreases total surplus and rent dissipation. The paper is organized as follows. In Section 2 we present the model. In Section 3 we characterize pairwise equilibrium networks. Section 4 analyzes efficiency and contrasts the efficient networks to the pairwise equilibrium networks. Section 5 concludes.

2. Model and notation

2.1. Networks

Let $N = \{1, 2, ..., n\}$ be the finite set of agents. Each agent announces the set of links she would like to form, and the links that form are those where both agents have announced their intention to form that link. We let $\sigma_{i,j} = 1$ if agent i intends to form a link with agent j while $\sigma_{i,j} = 0$ if she does not. The strategy of agent i is $\sigma_i = \{\{\sigma_{i,j}\}_{j \in N \setminus \{i\}}\} \in S_i$. When $\sigma_{i,j} = \sigma_{j,i} = 1$, a link between i and j is formed and is denoted by ij. A network $g = \{(g_{ij})\}$ is the list of pairs of individuals who are linked to each other. Let g^N be the collection of all subsets of N with cardinality 2, so g^N is the complete network. The set of all possible networks on N is denoted by $\mathbb{G}$ and consists of all subsets of g^N. A strategy profile $\sigma = (\sigma_i)_{i \in N}$ therefore induces a network $g(\sigma) \in \mathbb{G}$. The network obtained by adding the link ij to an existing network g is denoted $g + ij$ and the network that results from deleting the link ij from an existing network g is denoted $g - ij$. For any network g, let $N(g) = \{i \in N \mid \exists j \text{ such that } ij \in g\}$ be the set of agents who have at least one link in the network g. Let $N_i(g)$ be the set of agents who are linked to $i : N_i(g) = \{j \in N \mid ij \in g\}$. The degree of agent i in a network g is the number of links that involve that agent: $d_i(g) = \#N_i(g)$. A path in a network $g \in \mathbb{G}$ between i and j is a sequence of agents $i_1, \ldots, i_K$ such that $i_k i_{k+1} \in g$ for each $k \in \{1, \ldots, K-1\}$ with $i_1 = i$ and $i_K = j$. A network g

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movic (2014) and König et al. (2011).

is connected if for each pair of agents i and j such that $i \neq j$ there exists a path in g between i and j. A component h of a network g is a nonempty subnetwork $h \subseteq g$ satisfying (i) for all $i \in N(h)$ and $j \in N(h)\backslash\{i\}$, there exists a path in h connecting i and j, and (ii) for any $i \in N(h)$ and $j \in N(g)$, $ij \in g$ implies $ij \in h$. Given a set of s agents S, where $S \subsetneq N$, a *dominant group network* g^S is such that the agents in S are connected to each other and the agents in $N\backslash S$ have no links. We sometimes say that members in S are completely connected when every link is formed among these agents. We write $X\top_g Y$ if each agent in X is connected to each agent in Y in g. For all $t \geqslant 2$, let $K_t(g) = \{i \in N \mid d_i(g) \geqslant d_j(g)\ for\ all\ j \in N \backslash (K_s(g))_{s<t}\}$ be the set of agents with the highest degree among the agents who are not in $K_1(g), ..., K_{t-1}(g)$. A network g is a *nested split graph* with t classes if $K_1(g)\top_g K_r(g)$ for all $r \leqslant t - s + 1$. The agents in class 1 are connected to every connected agent, while the agents in class t are only connected to the agents in class 1. Similarly, the agents in class 2 are connected to every connected agent other than those in class t while the agents in class $t - 1$ are only connected to the agents in classes 1 and 2, etc.

2.2. How networks affect the Tullock contest

We consider a society of n ex-ante identical individuals who first form links of collaboration and then choose their level of effort in order to win a prize in a contest. The prize is allocated to the individuals according to the profile of efforts of all agents. We assume that agent i's probability of winning the prize, p_i, is given by the ratio of her effort (e_i) and the sum of all efforts $p_i(e_i, e_{-i}) = e_i / \sum_{j \in N} e_j$. The cost of providing effort is assumed to be equal to the level of effort. We are aware that this assumption is restrictive. A slightly more general assumption, however, would hamper the characterization of the expected payoffs of the contest stage in a tractable way and this would not allow us to study network formation. The valuation of agent i for the good is decomposed into a fixed component v and a variable component that depends on the degree of the agent $v_i(g) = v + d_i(g)\beta$. Both the fixed valuation v and the impact of a link on the valuation of an agent, β, is common to each agent and does not depend on the number of links the agent has. We restrict our analysis in this paper to the case where the cost of link formation c is positive but close to 0. Forming cross-licensing agreements, establishing joint R&D programs or sharing valuable information certainly involves costs in terms of organization, administration, layers, etc. However, we believe that in many applications we have discussed in the introduction the value of the prize is so important that these costs alone do not determine the partnership choices of the agents. Given a network g, the expected payoff of agent i is given by

$$\Pi_i(e_i, e_{-i}, g) = p_i(e_i, e_{-i})(v + d_i(g)\beta) - e_i - cd_i(g), \tag{1}$$

2.3. Equilibrium contest efforts

Given a network g, an agent i chooses a nonnegative effort e_i in order to maximize her payoff $\Pi_i(e_i, e_{-i}, g)$. We do not exclude corner solutions to this problem. There are network architectures and parameters configurations such

that the optimal effort of an agent is null at the Nash equilibrium of the contest stage. For this agent the participation constraint is binding. Let us introduce the ordering function $\phi : N \times \mathbb{G} \rightarrow N$ to rank the agents according to their valuation in the network so that the agents with a higher valuation have a lower index in the reordering. When multiple agents have the same valuation, the agents with lower indices in the agent set have a lower rank in the ranking. It follows that $\phi(k, g) = 1$ if $v_k(g) \geq v_l(g)$ for all $l \in N$ and $k < l$ for all l such that $v_k(g) = v_l(g)$.

From Hillman and Riley (1989), the number of participating agents is the largest integer $\phi(k, g)$ such that

$$v_k(g) > \frac{\phi(k, g) - 1}{\sum_{j:\phi(j,g) \leq \phi(k,g)} \frac{1}{v_j(g)}}.$$

We let $\kappa(g)$ be the value of $\phi(k, g)$ that solves this problem, and denote the set of agents participating to the contest in the network g by $K(g)$. Thus, the $\kappa(g)$ agents with the highest valuation at the network g participate in the contest, while the remaining agents do not. In particular, for k such that $\phi(k, g) \leq (>)\kappa(g)$, we have

$$v_k(g) > (\leq) \frac{(\kappa(g) - 1)}{\sum_{j \in K(g)} \frac{1}{v_j(g)}} = h_{\kappa(g)}(g) \frac{(\kappa(g) - 1)}{\kappa(g)},$$

where $h_{\kappa(g)}(g) = \frac{\kappa(g)}{\sum_{j \in K(g)} \frac{1}{v_j(g)}}$ is the harmonic mean[10] of the largest $\kappa(g)$ valuations in the network g.

Following Stein (2002, p. 329), in a network g, the Nash equilibrium level of effort is given by:

$$e_i^*(g) = \begin{cases} \frac{\kappa(g) - 1}{\kappa(g)} h_{\kappa(g)}(g) \left(1 - \frac{\kappa(g) - 1}{v_i(g)} \frac{h_{\kappa(g)}(g)}{\kappa(g)} \right) & \text{if } \phi(i, g) \leq \kappa(g) \\ 0 & \text{if } \phi(i, g) > \kappa(g) \end{cases} \quad (2)$$

so that the equilibrium probability that agent i wins the contest is

$$p_i^*(g) = \begin{cases} 1 - \frac{\kappa(g) - 1}{v_i(g)} \frac{h_{\kappa(g)}(g)}{\kappa(g)} & \text{if } \phi(i, g) \leq \kappa(g) \\ 0 & \text{if } \phi(i, g) > \kappa(g) \end{cases} \quad (3)$$

and the equilibrium payoff is

$$\Pi_i(e_i^*, e_{-i}^*, g) = \begin{cases} v_i(g) \left(p_i^*(g) \right)^2 - cd_i(g) & \text{if } \phi(i, g) \leq \kappa(g) \\ -cd_i(g) & \text{if } \phi(i, g) > \kappa(g) \end{cases} \quad (4)$$

When the number of participants is *fixed*, Stein (2002, Proposition 5, p. 391) shows that both the equilibrium probability of getting the prize and the equilibrium payoff of an agent is increasing in her valuation and is decreasing in the

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[10]The harmonic mean, h, of the positive real numbers $x_1, x_2, ..., x_n$ is defined to be $h = \frac{n}{\sum_{i=1}^{n} \frac{1}{x_i}}$.

valuation of another agent. In our framework, the agents have the ability to increase their valuation by forming links. However, when they create a new link, both their valuation and the valuation of a competitor increase so that the net effect on the payoff of the agent is unclear. In addition, by forming cooperative links, rivals may obtain an indirect benefit through the reduction of effort or the exclusion of less connected agents. We analyze the formation of networks of collaboration in the following section.

3. Pairwise equilibrium networks

For each network structure defining the profile of valuations of the competitors, there is a unique Nash equilibrium choice of efforts in the second stage. Solving the game by backward induction, we characterize the set of pairwise equilibrium networks of the link formation game.

Let $\Pi_i(e_i^*(g(\sigma)), e_{-i}^*(g(\sigma)), g(\sigma))$ be the payoff of agent i in the network $g(\sigma)$ induced by the strategy profile $\sigma = \{\sigma_1, \sigma_2, ..., \sigma_n\}$ when the agents choose their Nash equilibrium contest effort $e^*(g(\sigma)) = (e_1^*(g(\sigma)), ..., e_n^*(g(\sigma)))$ in the second stage. A strategy profile $\sigma^* = \{\sigma_1^*, \sigma_2^*, ..., \sigma_n^*\}$ is a Nash equilibrium of the link formation game if for all $\sigma_i \in S_i$ and for all $i \in N$:

$$\Pi_i \left(e_i^*(g(\sigma^*)), e_{-i}^*(g(\sigma^*)), g(\sigma^*) \right) \geq$$
$$\Pi_i \left(e_i^* \left(g(\sigma_i, \sigma_{-i}^*) \right), e_{-i}^* \left(g(\sigma_i, \sigma_{-i}^*) \right), g(\sigma_i, \sigma_{-i}^*) \right).$$

As is standard in the theory of network formation (Jackson and Wolinsky, 1996), we add to the notion of Nash equilibrium the requirement that there does not exist a pair of agents that would like to form a link. Pairwise equilibria are Nash equilibria satisfying this additional requirement. Let $\pi_i(g) = \Pi_i(e_i^*(g(\sigma)), e_{-i}^*(g(\sigma)), g(\sigma))$ for some σ such that $g = g(\sigma)$.

Definition 1. A network g is a **pairwise equilibrium** network if

1. there is a Nash equilibrium σ^* that induces g;
2. for all $ij \notin g$, if $\pi_i(g) < \pi_i(g + ij)$ then $\pi_j(g) > \pi_j(g + ij)$.

A game of network formation satisfies link monotonicity among participating agents if two participating agents are always better off by forming or maintaining a link between them.

Definition 2. A network formation game satisfies **link monotonicity among participating agents** if $\pi_i(g) < \pi_i(g+ij)$ and $\pi_j(g) < \pi_j(g+ij)$ for all $g \in \mathbb{G}$ and $i, j \in K(g)$ such that $ij \notin g$.

We show that the game does not satisfies link monotonicity among participating agents.

Lemma 1. *Link monotonicity among participating agents is not satisfied.*

Proof. The proof of this lemma and all subsequent proofs are to be found in the appendix. □

To establish this result, we provide an example where only two agents are participating, they are not connected among each other and one of these agents has more connections than the other. The agent with more connections may then prefer not to add the link that is missing between them. The monotonicity result of Stein (2002, Proposition 5, p. 391) by which an increase in the valuation of one agent increases her payoff does not apply to the case where the valuations of two agents increase by the same amount. Other models of network formation between competitors (Goyal and Joshi 2003, Westbrock 2010, Marinucci and Vergote 2011) satisfy link monotonicity when the costs of link formation are negligible, implying that the complete network is a pairwise equilibrium. Furthermore, it is the only one when participation constraints are not binding.

Even though two participating agents may not be both better off by forming a link, we show in Proposition 1 that two participating agents are connected in a pairwise equilibrium. Participating agents do cooperate.

Proposition 1. *Every network* $g \subseteq g^N$ *such that* $i, j \in K(g)$ *and* $ij \notin g$ *is not a pairwise equilibrium.*

We provide a short sketch of the proof, which is decomposed in 4 steps. In Step 1, we show that the set of participating agents in the network $g + ij$ is a subset of the set of participating agents in the network g. In Step 2, we show that the agents i and j participate in $g + ij$. In Step 3, we show that the agents i and j are better off by adding the link ij if the set of participating agents is the same under the networks g and $g + ij$. Finally, in Step 4, we show that if the set of participating agents is smaller under the network $g + ij$ than under g, then there always exist two agents k and l who participate to the contest given network g, who are not linked to each other and who are better off in $g + kl$."

In a pairwise equilibrium network, we thus have a group of agents who are connected among themselves and participate to the contest and another group of agents who are isolated and are better off by not participating to the contest. Hence the pairwise equilibrium networks are dominant group networks; g^S for some $S \subseteq N$. We characterize the set of pairwise equilibrium networks in Proposition 2.

Proposition 2. *A network* g *is a pairwise equilibrium if*

(i) $g = g^N$,

(ii) $g = g^S$ *for all* $S \subseteq N$ *such that* $\frac{3}{2} + \sqrt{\frac{5}{4} + \frac{v}{\beta}} \leq s \leq n - 2$,

(iii) $g = g^{N\setminus\{i\}}$ *for all* $i \in N$ *when*

  − *either*
$$\frac{(v + \beta)(n - 2)}{v + (n - 2)\beta} + \frac{v + \beta}{v + (n - 1)\beta} \leq n - 2;$$

$-$ *or* $n \geq \left(\frac{v}{\beta}\right)^{\frac{1}{2}} + 2$ *and*

$$\frac{v + (n-2)\beta}{(n-1)^2} > (v + (n-1)\beta)\left(1 - \frac{n-1}{1 + \frac{(n-2)(v+(n-1)\beta)}{v+(n-2)\beta} + \frac{(v+(n-1)\beta)}{v+\beta}}\right)^2.$$

Note that the complete network is always a pairwise equilibrium. If an agent deviates from the complete network by cutting links, she either reaches a network g' where she is not participating or where she is participating but not connected to some participating agents. In both cases she is better off by maintaining her links. A dominant group network is a pairwise equilibrium when two unconnected agents would not participate after adding a link between them. Group dominant networks are pairwise equilibria if the relative weight of partnerships in determining the valuation of agents for the prize is sufficiently high. The higher the relative importance of collaboration in determining the valuation, the smaller the dominant group can be in a pairwise equilibrium. Equally, the larger the size of the group in a dominant group network, the smaller the incentives of unconnected agents to participate because, as the size of the group increases, there are more competitors and they each have a higher valuation for the prize. When the size of the group is larger than the threshold value $(3/2 + \sqrt{5/4 + v/\beta})$, no pair of unconnected agents would participate by adding a link. It follows that if the population size is smaller than the threshold value, then the only pairwise equilibrium is the complete network. On the other hand, there are always pairwise equilibria leading to the exclusion of some agents in large populations. If the dominant group network g^S is a pairwise equilibrium, each dominant group network g^T is a pairwise equilibrium as long as $\#T \geq \#S$. If g^S is not a pairwise equilibrium, and the group S is composed of $n-2$ agents, then the condition to determine whether the network g^T that connects $n-1$ agents is a pairwise equilibrium changes. The network g^T is a pairwise equilibrium either if the isolated agent would not participate by adding a link with a participating agent, or if an agent from the group prefers not to add the link with the isolated agent.

4. Efficiency

In this section, we discuss the relationship between the network architecture and the total surplus. A network is efficient if it maximizes the sum of payoffs of the agents when they choose their Nash equilibrium effort in the second stage. Denote $W(g)$ as the sum of expected payoffs of all agents in network g : $W(g) = \sum_{i \in N} \pi_i(g)$. A network g is efficient if no other network would generate a higher sum of payoffs.

Definition 3. A network g is efficient if $W(g) \geq W(g')$ for all $g' \in \mathbb{G}$.

Notice that the total surplus of a network g, $W(g)$, is increasing in the expected valuation of the agent getting the prize and decreasing in total wasted efforts[11]:

$$W(g) = \sum_{i \in N} p_i^*(g) v_i(g) - \sum_{i \in N} (e_i^*(g) + d_i(g)c).$$

Since we assume that $c \simeq 0$, we will ignore the impact of the cost of link formation c on total surplus in what follows. Inefficiencies may arise because not all network benefits are exploited, because the prize is allocated with some probability to agents who do not have the maximal valuation for the good, and because resources are wasted to influence the allocation of the prize. Using (3), the sum of expected valuations is given by

$$\sum_{i \in N} p_i^*(g) v_i(g) = \sum_{i \in K(g)} v_i(g) - (\kappa(g) - 1) h_{\kappa(g)}(g).$$

Using (2), the sum of efforts is given by

$$\sum_{i \in K(g)} e_i^*(g) = e^*(g) = (\kappa(g) - 1) h_{\kappa(g)}(g) / \kappa(g);$$

which is increasing in the harmonic mean of the valuation of the participating agents. We thus obtain that

$$W(g) = \sum_{i \in K(g)} v_i(g) - h_{\kappa(g)}(g) \frac{(\kappa(g))^2 - 1}{\kappa(g)}. \tag{5}$$

It follows from (5) that for a fixed number of links and given a set of participating agents, the higher the harmonic mean of the valuations of the participating agents, the higher the sum of efforts and the lower the expected payoff of the agent getting the prize. If the set of participating agents is equal in two networks g and g' and the valuation profile of the participating agents in the network g' is a mean preserving spread of the valuation profile of the participating agents in the network g, then the sum of expected payoffs is higher under the network g' than under the network g.[12] In Proposition 3, we show that the total surplus increases (and the sum of agents' efforts decreases) when the distribution of the agents' valuations becomes more asymmetric, whenever the set of participating agents does not shrink.

Proposition 3. *Let $g, g' \in \mathbb{G}$ be such that $d_i(g) \geq d_j(g)$, $d_i(g') = d_i(g) + 1$, $d_j(g') = d_j(g) - 1$ and $d_l(g) = d_l(g') \ \forall l \neq i, j$ such that $j \in K(g')$. Then $W(g') > W(g)$ and $\Sigma_{l \in N} e_l^*(g') < \Sigma_{l \in N} e_l^*(g)$.*

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[11]The definition of total wasted efforts is equivalent to rent dissipation: the total expenditure on effort by all agents who are competing to obtain the prize. Agents could jointly do better by reducing their efforts by the same percentage. Such reduction would not change the probability of winning of each agent, but decrease her effort level. However, it would also create incentives for unilateral deviations.

[12]As mentioned above, this result was established by Stein (2002, Proposition 2, p. 390).

If a central planner who aims at maximizing total surplus could reallocate the existing links of some network g, he should reallocate them in order to increase the asymmetry among agents, as long as the set of active participants does not shrink by doing so. As a corollary to Proposition 3, we have that the efficient network is a nested-split graph when participation constraints are not binding. Nested-split graphs are such that it is not possible to reallocate links in a way that reinforces asymmetry.

Corollary 1. *Let $K(g) = N$ for all $g \in \mathbb{G}$. Then the efficient network is a nested split graph.*

Total surplus does not necessarily increase when some agent leaves the contest after the reallocation of links, as stated in Proposition 4.

Proposition 4. *Let $g, g' \in \mathbb{G}$ be such that $d_i(g) \geq d_j(g)$, $d_i(g') = d_i(g) + 1$, $d_j(g') = d_j(g) - 1$ and $d_l(g) = d_l(g') \; \forall l \neq i, j$ such that $j \notin K(g')$. Then $W(g') \geqslant W(g)$ or $W(g') < W(g)$.*

Two examples show that Proposition 4 must hold[13]. In both examples, the network g' is obtained from g replacing a link by another such that (i) one agent has one more link in g' than in g while another has one link less, (ii) the agent whose degree decreases has initially no more links than the agent whose degree increases, and (iii) the agent whose degree decreases does no longer participate to the contest after the reallocation of the link. In the first example, the total surplus decreases after the redistribution of links while in the second it increases. In both examples, let $n = 11$. Moreover, suppose the common valuation for the good is $v = 0$ and the effect of cooperation on the valuation is $\beta = 1$. In the first example, a reallocation of a link decreases the participants and total surplus.

Example 1. Let the network g be such that agent 1 is connected to all agents, agent 2 is connected to agents 1, 3, and 4, and agent 3 is connected to agents 1, 2 and 5. The valuation of contestants for the prize is then given by $v_1 = 10, v_2 = 3, v_3 = 3$ and $v_k \leq 2$ for $k \geq 4$. Then, agents 1, 2 and 3 participate and get respectively a payoff of $\pi_1(g) = 5.46$, $\pi_2(g) = 0.05$, and $\pi_3(g) = 0.05$. Let the network g' be obtained from g by replacing the link $i_3 i_5$ by the link $i_2 i_5$. Agent 3 no longer participates in g' and payoffs are $\pi_1(g') = 5.10$ and $\pi_2(g') = 0.33$ so that the total surplus decreases. The sum of efforts increases from $\sum_{i \in N} e_i^*(g) = 2.61$ to $\sum_{i \in N} e_i^*(g') = 2.86$, and the probability agent 1 gets the prize falls from 0.74 in g to 0.71 in g'.

In the second example, a reallocation of a link decreases the participants and increases total surplus.

Example 2. Let the network g be such that agent 1 is connected to 2, 3, 8, 9, 10, agent 2 is connected to 1, 4 and 5 and agent 3 is connected to agents 1 and 6. The valuation of contestants for the prize is then given by $v_1 = 5, v_2 = 3, v_3 = 2$

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[13]Since these examples are instructive for the discussion we present them in the main text.

and $v_k \leq 1$ for $k \geq 4$. Then, agents $1, 2$ and 3 participate and get respectively a payoff of $\pi_1(g) = 1.88$, $\pi_2(g) = 0.38$, and $\pi_3(g) = 0.002$. Let the network g' be obtained from g by replacing the link $i_3 i_6$ by the link $i_2 i_6$. Agent 3 no longer participates in g' and payoffs are $\pi_1(g') = 1.54$ and $\pi_2(g') = 0.79$ so that the total surplus increases. The sum of efforts increases from $\sum_{i \in N} e_i^*(g) = 1.94$ to $\sum_{i \in N} e_i^*(g') = 2.22$, and the probability agent 1 gets the prize falls from 0.61 to 0.56.

The agent who leaves the contest after the reallocation of links was exerting relatively little effort before the reallocation of links. The reallocation of links is then almost equivalent to reinforcing the valuation of one single agent. This could harm total surplus if the distribution of the valuations of the active contestants becomes more equal. As the competition becomes more equal, more resources are wasted in the contest, and the prize is allocated with a higher probability to the agent whose degree has increased. This induces allocative inefficiency if the valuation of this agent is small relative to the others. The payoff of this agent increases not only because she gets the prize with a higher probability, but also because her valuation has increased. As a consequence, the reallocation of links may have a positive or negative impact on total surplus, depending on the magnitude of the effects described. Cornes and Hartley (2005, Proposition 6, p. 940) show that limiting the number of participants or increasing the asymmetry between agents decreases rent dissipation. We qualify this result. When participation constraints matter, rent dissipation may increase when the social planner reallocates links in a way that strengthens the asymmetry between agents and reduces the number of active agents.

We now discuss the relationship between stability and efficiency. We show in Lemma 2 that the pairwise equilibria can be ranked in terms of total surplus. In smaller dominant group networks, the valuation of the agent getting the prize is smaller but the sum of wasted efforts is also reduced since fewer agents compete for the prize and they value the prize less. When the common valuation of agents for the good is important relative to the network benefits ($v/\beta > 1$), smaller dominant group networks produce more surplus than larger ones. Endogenous barriers to entry enhance total surplus in this case. If the valuation of agents for the prize mainly depends on their connections in the network ($v/\beta < 1$), then the sum of payoffs increases with the size of the dominant group network. Endogenous barriers to entry then hurt total surplus.

Lemma 2. $W(g^S) > W(g^T)$ *when $s > t$ and $K(g^T) = T$ if and only if $v/\beta < 1$.*

We show in Proposition 5 that a pairwise equilibrium is not efficient whenever the complete network is not the only pairwise equilibrium network.

Proposition 5. *Every pairwise equilibrium $g^S \neq g^N$ is not efficient. In addition, g^N is not efficient if $g^S \neq g^N$ is a pairwise equilibrium.*

If the valuation of agents for the prize mainly depends on their connections in the network ($v/\beta < 1$), the complete network is the pairwise equilibrium that also maximizes the sum of payoffs. Even if the prize is allocated to an agent with the maximal possible valuation in the complete network, it generates less surplus than the star network. In the star network, the gap between the valuation of the center of the star and the valuation of the other agents is so important that most of the effort is done by the center of the star. She gets the prize with high probability while the sum of efforts is relatively small. If, on the other hand, the valuation of the agents for the prize mainly depends on their fixed valuation for the good ($v/\beta > 1$), the pairwise equilibrium network that generates the highest sum of payoffs is the one with the smallest group of completely connected agents g^S. If g^S is not the complete network, it is not efficient since the sum of payoffs is higher in the adjacent network $g^S + ij$ where the link added involves an agent in the group and another outside the group. In a model of cost-reducing network formation in an oligopolistic market, Westbrock (2010) finds that efficient networks must either be dominant group or have the interlinked star architecture.[14] In the Tullock contest, we show that pairwise equilibrium dominant group networks other than the complete network never maximize total surplus.

5. Conclusion

Following Hillman and Riley (1989), Stein (2002), and Cornes and Hartley (2005), we have analyzed a Tullock contest among asymmetric agents when participation is not binding. In our model, the asymmetry between the agents is endogenously determined by the cooperative links they have with their competitors. We have established that the only pairwise equilibrium networks are dominant group networks. Agents in the dominant group are completely connected to each other while those outside the group are not connected and do not participate. Network formation thus acts as a barrier to entry to the contest. We have then analyzed the impact of network formation on total surplus. We have found that a social planner may increase the total surplus by reallocating links in order to increase the asymmetry between agents, when this policy does not reduce the number of participating agents. This is not necessarily true if fewer contestants participate. We have shown that barriers to entry may either hurt total surplus as the winner of the prize does not exploit all the possible network benefits, or improve it since less rent is dissipated as competition becomes less fierce.

Marinucci and Vergote (2011) were the first to shed light on the importance of network formation to explain barriers to entry. While focusing on an all pay auction, their result rests on the assumption that the valuation of agents is increasing exponentially in the amount of links they have. We study network

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[14]In an interlinked star network, a group of agents (the center) is connected to each agent with at least one link.

formation in the Tullock contest instead and assume that links affect the valuation of agents in a linear fashion. In addition, we analyze in detail the role of network formation on efficiency, and the impact of barriers to entry on total surplus. In this respect, we complement the work of Westbrock (2010) who has analyzed efficiency in a setting where cooperative R&D links reduce the marginal cost of firms competing in an oligopolistic setting. Contrary to Westbrock (2010), we do not assume that agents always wish to participate to the competition, which complicates the analysis and brings about new questions. While we have not been able to fully characterize the efficient networks, we have nonetheless shown that when the participation constraints are not binding, the efficient network must be a nested split graph. In order to analyze the role of collaboration through link formation in the Tullock contest we have consciously chosen a very simple framework in which links affect the valuation of agents linearly, contest effort functions are also linear and there are no linking costs. This leaves ample room for generalizations and alternative specifications to tackle questions that remain unanswered. It would be interesting to analyze the effect of ex-ante heterogeneity. Would one still predict only dominant group networks to arise in equilibrium? In reality we also often observe several groups of agents who cooperate amongst themselves, while competing with one another. Is there a way to adjust the assumptions of the model to obtain the emergence of teams in equilibrium? Would one observe this if there are several prizes? We leave these questions for future research.

Acknowledgements

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Appendix

Proof of Lemma 1.

Let $v = \beta = 1$ and hence $v/\beta = 1$. Let g be such that $d_1(g) = 9$, $d_2(g) = 2$, $d_i(g) \leq 1$ for all $i \neq 1, 2$ and $g_{12} = 0$. Then we have that $K(g) = \{1, 2\}$ and $\pi_1(g) = \frac{1000}{169}$. We have $K(g+12) = \{1, 2\}$ and $\pi_1(g+12) = \frac{1331}{225} < \frac{1000}{169} = \pi_1(g)$. It follows that agent 1 does not want to form the link 12.□

Before the proofs of Propositions 1, 2 and 5 we introduce an auxiliary lemma, used to prove the proposition. We label these lemmas A1, A2 and A3 respectively. Lemma A1 establishes that the harmonic mean of the valuation of the x agents with the highest valuation, weighted by $(\frac{x-1}{x})$ is a function that is increasing in x as long as x is smaller than the number of participating agents.

Lemma A1. For all $x \in \{1, 2, ..., \kappa(g) - 1\}$, we have

$$\frac{x-1}{x} h_x(g) < \frac{x}{x+1} h_{x+1}(g) \tag{1A}$$

Proof of Lemma A1.

Let $x \in \{1, 2, ..., \kappa(g) - 1\}$. Let agent l be such that $\phi(l, g) = x + 1$. After some simple calculations, we may rewrite $(1A)$ as

$$v_l(g) > \frac{x-1}{x} h_x(g) \tag{2A}$$

Let agent k be such that $\phi(k, g) = \kappa(g)$. The participation constraint of agent k may be written as

$$\sum_{i \in K(g)} \frac{v_k(g)}{v_i(g)} > \kappa(g) - 1,$$

$$\sum_{i \in N: 1 \leq \phi(i,g) \leq x} \frac{v_k(g)}{v_i(g)} + \sum_{j \in N: x+1 \leq \phi(j,g) \leq \kappa(g)} \frac{v_k(g)}{v_j(g)} > \kappa(g) - 1,$$

Since $v_k(g) \leq v_j(g)$ for all $j \in K(g)$, we have $\sum_{j \in N: x+1 \leq \phi(j,g) \leq \kappa(g)} \frac{v_k(g)}{v_j(g)} \leq \kappa(g) - x$. We thus have $\sum_{i \in N: 1 \leq \phi(i,g) \leq x} \frac{v_k(g)}{v_i(g)} > x - 1$, or $v_k(g) > \frac{x-1}{x} h_x(g)$. It follows that $(2A)$ is satisfied since $v_l(g) \geq v_k(g)$.□

Proof of Proposition 1.

Let $g \subseteq g^N$ with $ij \notin g$ for some $i, j \in K(g)$. On the contrary, suppose g is a pairwise equilibrium network. Suppose first that $d_k(g) > 0$ for some $k \notin K(g)$, then agent k would be better off by deleting her links, a contradiction. It follows that $\kappa(g) \geq 3$. Otherwise, we would have $i, j \in K(g)$, $ij \notin g$ and $d_k(g) = 0$ for all $k \in N \backslash \{i, j\}$ so that g would be the empty network. But then, we would have $K(g) = N$, a contradiction. In the rest of the proof, we consider the network g such that $\kappa(g) \geq 3$ and $d_k(g) = 0$ for all $k \in N \backslash K(g)$. We decompose the proof into 4 steps.

Step 1. $K(g+ij) \subseteq K(g)$.

Notice that $h_x(g+ij) \geq h_x(g)$ for all $x \in \{1, 2, ..., n\}$, a property of the comparison between (harmonic) means of two series of numbers $((v_k(g+ij))_{k \in N}$ and $(v_k(g)_{k \in N}$ where the first series has higher numbers than the second series. Since $i, j \in K(g)$, $v_k(g) = v_k(g+ij)$ for all $k \notin K(g)$. It follows that the no participation of agent k under the network g implies the no participation of agent k under the network $g+ij$: $v_k(g) < h_{\kappa(g)}(g)(\kappa(g)-1)/\kappa(g)$ implies $v_k(g+ij) < h_{\kappa(g)}(g+ij)(\kappa(g)-1)/\kappa(g)$ for all $k \notin K(g)$. Thus, $K(g+ij) \subseteq K(g)$.

Step 2. $i, j \in K(g+ij)$.

Suppose first that $i, j \notin K(g+ij)$. We then have $v_i(g+ij) > v_i(g) > \frac{\kappa(g)-1}{\sum_{l \in K(g)} \frac{1}{v_l(g)}} > \frac{\kappa(g+ij)-1}{\sum_{l \in K(g+ij)} \frac{1}{v_l(g)}} = \frac{\kappa(g+ij)-1}{\sum_{l \in K(g+ij)} \frac{1}{v_l(g+ij)}}$, where the second inequality is the participation constraint of agent i in the network g, the third holds by Lemma A1 since $\kappa(g) \geq \kappa(g+ij)$ and the last equality holds since $v_k(g+ij) = v_k(g)$ for all $k \in K(g+ij)$. We thus have a contradiction since we have assumed $i \notin K(g+ij)$ while $v_i(g+ij) > \frac{\kappa(g+ij)-1}{\sum_{l \in K(g+ij)} \frac{1}{v_l(g+ij)}}$. Suppose next that $\{i, j\} \cap K(g+ij) \neq \emptyset$ and $\{i, j\} \nsubseteq K(g+ij)$. Without loss of generality, suppose $i \notin K(g+ij)$ and $j \in K(g+ij)$, then

$$
\begin{aligned}
v_i(g+ij) \sum_{l \in K(g+ij)} \frac{1}{v_l(g+ij)} &> v_i(g) \sum_{l \in K(g+ij)} \frac{1}{v_l(g)} \geq \\
v_i(g) \sum_{\substack{l \in N: \\ \phi(l,g) \leq \kappa(g+ij)}} \frac{1}{v_l(g)} &> \kappa(g+ij) - 1
\end{aligned}
$$

where the first inequality holds since $i \notin K(g+ij)$ and $j \in K(g+ij)$ imply $v_i(g) < v_j(g)$, which in turn leads to $\frac{v_i(g+ij)}{v_j(g+ij)} > \frac{v_i(g)}{v_j(g)}$, the second inequality holds since the $\kappa(g+ij)$ agents whose valuation are considered under the operator $\phi(.,g)$ are those with the highest valuation in the network g, and the last inequality holds since $v_i(g) > \frac{\kappa(g)-1}{\kappa(g)} h_{\kappa(g)}(g) > \frac{\kappa(g+ij)-1}{\kappa(g+ij)} h_{\kappa(g+ij)}(g)$ by the participation constraint of agent i in the network g and Lemma A1. Since $v_i(g+ij) \sum_{l \in K(g+ij)} \frac{1}{v_l(g+ij)} > \kappa(g+ij) - 1$, we have $i \in K(g+ij)$, a contradiction. We have shown that we cannot have $i, j \notin K(g+ij)$, nor $\{i, j\} \cap K(g+ij) \neq \emptyset$ and $\{i, j\} \nsubseteq K(g+ij)$. Thus, $i, j \in K(g+ij)$.

Step 3. $\pi_i(g+ij) > \pi_i(g)$ and $\pi_j(g+ij) > \pi_j(g)$ if $K(g+ij) = K(g)$.

We show hereafter that $p_i^*(g+ij) - p_i^*(g) > 0$ both when $v_i(g) > v_j(g)$ and when $v_i(g) \leq v_j(g)$. As a consequence, we conclude that $p_j^*(g+ij) - p_j^*(g) > 0$, and that $\pi_k(g+ij) > \pi_k(g)$ for $k = i, j$.

Using $\kappa(g+ij) = \kappa(g) = \kappa$, notice that $p_i^*(g+ij) - p_i^*(g) = \frac{\kappa-1}{\kappa} \left(\frac{h_\kappa(g)}{v_i(g)} - \frac{h_\kappa(g+ij)}{v_i(g+ij)} \right)$. Thus $p_i^*(g+ij) - p_i^*(g) > 0$ if $h_\kappa(g)/v_i(g) > h_\kappa(g+ij)/v_i(g+ij)$, that is if

$$
\sum_{k: \phi(k,g) \leq \kappa(g)} \frac{v_i(g+ij)}{v_k(g+ij)} > \sum_{k: \phi(k,g) \leq \kappa(g)} \frac{v_i(g)}{v_k(g)}. \tag{3A}
$$

Using $v_k(g+ij) = v_k(g)$ for $k \neq i, j$ and $v_k(g+ij) = v_k(g) + \beta$ for $k = i, j$, and noting that

$$
\frac{\sum_{k: \phi(k,g) \leq \kappa(g)} \frac{1}{v_k(g+ij)}}{\sum_{k: \phi(k,g) \leq \kappa(g)} \frac{1}{v_k(g)}} + \frac{1}{v_i(g+ij)} + \frac{1}{v_j(g+ij)} - \frac{1}{v_i(g)} - \frac{1}{v_j(g)},
$$

let us rewrite Condition (A3) as

$$\beta \sum_{k:\phi(k,g)\leq\kappa} \frac{1}{v_k(g)} - \beta\left(\frac{1}{v_i(g)} + \frac{1}{v_j(g)}\right) > \frac{v_i(g)}{v_j(g)} - \frac{v_i(g)+\beta}{v_j(g)+\beta}. \tag{4A}$$

Notice that the left-hand side of Condition (4A) (hereafter LHS) is positive while the right-hand side (RHS) is negative when $v_i(g) \leq v_j(g)$. Let us show that Condition (4A) is also satisfied when $v_i(g) > v_j(g)$. From the participation constraint of agent j in the network g, we know that $v_j(g) > (\kappa(g) - 1)/\sum_{k\in K(g+ij)}\frac{1}{v_k(g)}$. Notice that since $v_i(g) > v_j(g)$, RHS is decreasing in $v_j(g)$ while LHS does not depend on $v_j(g)$. We show that Condition (4A) holds even if $v_j(g) = (\kappa(g)-1)/\sum_{k\in K(g+ij)}\frac{1}{v_k(g)}$. Indeed, we then have

$$\text{LHS} = \frac{\beta(\kappa(g)-1)}{v_j(g)} - \frac{\beta}{v_i(g)} - \frac{\beta}{v_j(g)} > \frac{\beta(\kappa(g)-3)}{v_j(g)} > \frac{\beta(v_i(g)-v_j(g))}{v_j(g)(v_j(g)+\beta)} = \text{RHS}$$

where the first inequality holds since $v_i(g) > v_j(g)$, while the second holds if

$$(\kappa(g)-2)v_j(g) + (\kappa(g)-3)\beta \geq v_i(g). \tag{5A}$$

We show that condition (5A) holds. Suppose that $d_j(g) \geq 1$. Since $ij \notin g$, and $d_k(g) = 0$ for all $k \notin K(g)$, we have $d_i(g) \leq \kappa(g) - 2$. It then follows that $v_i(g) - v_j(g) \leq (\kappa(g)-3))\beta$ implying that condition (5A) is satisfied since $\kappa(g) \geq 3$. If on the other hand $d_j(g) = 0$, then, since $j \in K(g)$, it follows that $\kappa(g) = n$, and condition (5A) becomes $(n-3)v + (n-3-d_i(g))\beta \geq 0$, which is satisfied when $d_i(g) \leq n-3$. When $d_i(g) = n-2$, the condition becomes

$$(n-3)v \geq \beta. \tag{6A}$$

Notice that $d_k(g) \geq 1$ for each agent $k \neq j$ since $d_j(g) = 0$ and $d_i(g) = n-2$. The participation constraint of agent j in the network g then implies that $n-1 < \sum_{k\in N}\frac{v}{v_k(g)} \leq 1 + \frac{v}{v+(n-2)\beta} + \frac{(n-2)v}{v+\beta} < 1 + \frac{(n-1)v}{v+\beta}$. We thus find that $v > (n-2)\beta$, establishing that condition (6A) holds as long as $n \geq 4$. When $N = \{i,j,k\}$ so that $n = 3$, the only network g such that $d_i(g) = n-2$ and $d_j(g) = 0$ is $g = \{ik\}$. In that case, $p_i^*(g+ij) > p_i^*(g)$ if $v > \beta$, which is satisfied as otherwise agent j would not participate in the contest in the network g.

Step 4. If $K(g+ij) \subsetneq K(g)$, then there always exists a pair of agents $k,l \in K(g)$ such that $\pi_k(g+kl) > \pi_k(g)$ and $\pi_l(g+kl) > \pi_l(g)$.

Step 4.a. Suppose that $\kappa(g+ij) \geq 3$. We show that $\pi_i(g+ij) > x > \pi_i(g)$, where

$$x = v_i(g)\left(1 - \frac{(\kappa(g+ij)-1)\,h_{\kappa(g+ij)}(g)}{\kappa(g+ij)v_i(g)}\right)^2.$$

Indeed, from Lemma A1, we have $x > \pi_i(g)$, while from step 3 in this proof, we know that $\pi_i(g+ij) > x$ when $\kappa(g+ij) \geq 3$. Following the same argument, we have $\pi_j(g+ij) > \pi_j(g)$.

Step 4.b. Suppose that $K(g + ij) = \{i, j\}$. Without loss of generality, suppose that $d_i(g) \geq d_j(g)$. From $K(g + ij) = \{i, j\}$, we know that there is no agent with more links than agent i in the network g, since this agent would otherwise participate in the network $g + ij$. Let l be an agent with the lowest degree among the participating agents, $l \in \{k \in K(g) \mid d_k(g) \leq d_m(g)$ for all $m \in K(g)\}$. There is a participating agent $k \in K(g)$ who is not connected to agent l; $kl \notin g$, since agent l does not have more links than agent i who is not connected to agent j and to non-participating agents in the network g. If $\kappa(g + kl) \geq 3$, then $\pi_k(g + kl) > \pi_k(g)$ and $\pi_l(g + kl) > \pi_l(g)$ by step 3 or 4.a. If, on the other hand $K(g + kl) = \{k, l\}$, then agent l has at least the same degree as agent i in the network g, meaning that agent l is not only the agent with the lowest degree among the participating agents in the network g, but also the agent with the highest degree. Thus each participating agent $m \in K(g)$ in the network g has the same number of links, so that her payoff is given by $\pi_m(g) = v_m(g)/(\kappa(g))^2$. It then follows that agent i and j (or any other pair of unconnected agents) are better off by forming a link in the network g as $\pi_m(g + ij) = (v_m(g) + \beta)/4 > \pi_m(g) = v_m(g)/(\kappa(g))^2$, for $m = i, j$.□

In order to prove Proposition 2, we introduce Lemma A2. First, it is shown that the isolated agents of the dominant group network g^S participate if and only if the size of the group S, denoted by s, is smaller than $(v/\beta)^{1/2} + 1$. Second, if only the members of S participate in the dominant group network g^S, then, by forming a link, two isolated agents do not see their valuation increase sufficiently in order to participate in the contest as long as the size of the group is greater than the threshold $3/2 + \sqrt{5/4 + v/\beta}$. Finally, it is shown that agents do not have incentives to delete links from a dominant group network where the unconnected agents do not participate.

Lemma A2. Let $S \subseteq N$
(i) $K(g^S) = \begin{cases} S & \text{if } s \geq (v/\beta)^{1/2} + 1; \\ N & \text{if } s < (v/\beta)^{1/2} + 1. \end{cases}$
(ii) Suppose $(v/\beta)^{1/2} + 1 \leq s$ so that $K(g^S) = S$, and let $k, l \in N\backslash S$. Then
$K(g^S + kl) = \begin{cases} S & \text{if } 3/2 + \sqrt{5/4 + v/\beta} \leq s; \\ S \cup \{k, l\} & \text{if } s < 3/2 + \sqrt{5/4 + v/\beta}. \end{cases}$
(iii) Suppose $(v/\beta)^{1/2} + 1 \leq s$ so that $K(g^S) = S$, then $\pi_i(g^S) > \pi_i(g')$ for $i \in S$, for all $g^{S\backslash\{i\}} \subseteq g' \subsetneq g^S$.

Proof of Lemma A2.
(i) $K(g^S) = S$ if for $k \in N\backslash S$, we have $v_k(g^S) \leq \frac{n-1}{n} h_n(g^S)$, that is if $v \leq \frac{s-1}{s/(v+(s-1)\beta)}$, or if $s \geq (v/\beta)^{1/2} + 1$. Otherwise if $s < (v/\beta)^{1/2} + 1$, then $v_k(g^S) > \frac{n-1}{n} h_n(g^S)$ so that every agent gets a positive payoff by participating.

(ii) Given $K(g^S) = S$, we have $K(g^S + kl) = S$ if $v_k(g^S + kl) \leq \frac{s-1}{s} h_s(g^S + kl)$ and $K(g^S + kl) = S \cup \{k, l\}$ otherwise.

(iii) Let us write $g' = g^S - ij_1 - ... - ij_q$ where $j_1, ..., j_q \in S$. Notice that $K(g') \in \{S\backslash\{i, j_1, ..., j_q\}, S\backslash\{i\}, S, N\}$. The result $\pi_i(g^S) > \pi_i(g')$ holds if

$i \notin K(g')$ since agent i has a positive payoff only if she participates. Also, if $s = 2$, we have $\pi_i(g^S) = (v + \beta)/4 > \pi_i(g^S - ij_1) = v/n^2$. In the rest of the proof, we consider the cases where $s \geq 3$ and $K(g') \in \{S, N\}$. Let $g_0 = g^S$, $g_1 = g^S - ij_1$, $g_2 = g_1 - ij_2$, ..., $g_q = g_{q-1} - ij_q$. By Step 2 of Proposition 1, we know that $i \in K(g_{q-1})$ since $i \in K(g_q)$ and $g_{q-1} = g_q + ij_q$. By repeating this argument, we find that $i \in K(g_p)$ for all $p = 1, ..., q$. Since $d_i(g_p) \leq d_m(g_p)$ for all $m \in S$, for all $p = 1, ..., q$, we have $S \subseteq K(g_p)$. Agents outside S have the same valuation so that either they all participate in a network in the sequence or none of them participates: $K(g_p) \in \{S, N\}$ for $p = 1, ..., q$. From Step 1 of Proposition 1, we know that $K(g_0) \subseteq K(g_1) \subseteq ... \subseteq K(g_q)$. Consider two adjacent networks in the sequence, say g_p and g_{p+1}. Either $K(g_p) = K(g_{p+1})$ so that $\pi_i(g_p) > \pi_i(g_{p+1})$ by Step 3 of Proposition 1, or $K(g_p) = S$ while $K(g_{p+1}) = N$ so that $\pi_i(g_p) > \pi_i(g_{p+1})$ by Step 4.a of Proposition 1. It follows that $\pi_i(g_0) > \pi_i(g_1) > ... > \pi_i(g_q).\square$

Proof of Proposition 2.
First, let

$$
\begin{cases}
E_1 = & \frac{3}{2} + \sqrt{\frac{5}{4} + \frac{v}{\beta}}; \\
E_2 = & \frac{(v+\beta)(n-2)}{v+(n-2)\beta} + \frac{(v+\beta)}{(v+(n-1)\beta)}; \\
E_3 = & (v/\beta)^{1/2} + 2; \\
E_4 = & \frac{(v+(n-2)\beta)}{(n-1)^2}; \\
E_5 = & (v + (n-1)\beta)\left(1 - \frac{(n-1)}{1 + \frac{(n-2)(v+(n-1)\beta)}{v+(n-2)\beta} + \frac{(v+(n-1)\beta)}{v+\beta}}\right)^2.
\end{cases}
$$

From Proposition 1, $g = g^S$ and $K(g) = S$ for some $S \subseteq N$ are necessary conditions for g to be a pairwise equilibrium network. From any such network, link deletion is not profitable by part (iii) of Lemma A2. Then a necessary and sufficient condition for g to be a pairwise equilibrium network is that $g = g^S$, $K(g) = S$ and for all $ij \notin g$, we have $\pi_i(g) > \pi_i(g + ij)$ or $\pi_j(g) > \pi_j(g + ij)$.

(i) g^N is a pairwise equilibrium since $K(g^N) = N$ and link addition is not feasible.

(ii) Let g^S for some $S \subseteq N$ be such that $E_1 \leq s \leq n - 2$. By part (i) of Lemma A2, we have $K(g^S) = S$ since $(v/\beta)^{1/2} + 1 \leq E_1 \leq s$. Notice that $\pi_k(g^S + kl) = -c$ for $k, l \notin S$ if $v + \beta \leq \frac{(s-1)}{\sum_{j \in S} 1/v_j(g)}$, while $\pi_k(g^S + ik) = -c$ for $i \in S$, $k \notin S$ if $v + \beta \leq \frac{(s-1)}{\sum_{j \in S} 1/v_j(g+ik)}$. Thus, link addition is not profitable if two agents without links in g^S are not better off by adding a link, i.e. $E_1 \leq s$ (see part (ii) of Lemma A2).

(iii) Take $g^{N\setminus\{i\}}$ for some $i \in N$. We show that $g^{N\setminus\{i\}}$ is a pairwise equilibrium network if either $K(g^{N\setminus\{i\}} + ij) = N\setminus\{i\}$ or if $K(g^{N\setminus\{i\}}) = N$ and $\pi_j(g^{N\setminus\{i\}} + ij) < \pi_j(g^{N\setminus\{i\}})$. If $E_2 \leq n - 2$, we have $K(g^{N\setminus\{i\}} + ij) = N\setminus\{i\}$ so that agent i is better off by not adding the link. If

$E_2 > n - 2$ so that $K(g^{N\backslash\{i\}} + ij) = N$, then two cases should be considered. First suppose that $n \geq E_3$ so that $K(g^{N\backslash\{i\}}) = N\backslash\{i\}$, then $\pi_j(g^{N\backslash\{i\}} + ij) < \pi_j(g^{N\backslash\{i\}})$ if $E_4 \geq E_5$. Second, suppose that $n < E_3$ so that $K(g^{N\backslash\{i\}}) = N$. Then, the network $g^{N\backslash\{i\}}$ is not a pairwise equilibrium network by Proposition 1.$\square$

Proof of Proposition 3.

(i) Note first that such a mean preserving spread cannot lead to fewer participants in g' compared to $g : K(g) \subseteq K(g')$ since $j \in K(g')$, and $\frac{(\kappa(g)-1)}{\kappa(g)} h_{\kappa(g)}(g) > \frac{(\kappa(g)-1)}{\kappa(g)} h_{\kappa(g)}(g')$.

(ii) Now we show that $e^*(g) > e^*(g')$:

(ii.a) If $K(g) = K(g')$, the result holds since the harmonic mean of the valuations of participating agents is lower in g' than in g.

(ii.b) If $K(g) \subsetneq K(g')$, the result holds since for $j \in K(g')\backslash K(g)$, we have $e^*(g) \geq v_j(g) > e^*(g')$.

(iii) We end by showing that $W(g') > W(g)$:

(iii.a) If $K(g) = K(g')$, then this holds by (ii).

(iii.b) If $K(g) \subseteq K(g')$, then

$W(g') - W(g) = \sum_{j \in K(g')\backslash K(g)} v_j(g) - (\kappa(g') + 1)e^*(g') + (\kappa(g) + 1)e^*(g)$.
$\qquad = \sum_{j \in K(g')\backslash K(g)} v_j(g) - (\kappa(g') - \kappa(g))e^*(g') + (\kappa(g)+1)(e^*(g) - e^*(g'))$

For all $j \in K(g')\backslash K(g)$, we have $v_j(g') > e^*(g')$ by the participation constraint of j. Thus, $\sum_{j \in K(g')\backslash K(g)} v_j(g) - (\kappa(g') - \kappa(g))e^*(g') > 0$ and the result then holds since $(e^*(g) - e^*(g')) > 0$.$\square$

Proof of Proposition 4.

See the main text. $\square$

Proof of Lemma 2.

The total surplus in a dominant group network g^X where $X \subseteq N$ and $K(g^X) = X$ is given by

$$W(g^X) = \frac{v}{x} + \frac{(x-1)}{x}\beta$$

Notice that $K(g^T) = T$ and $t < s$ imply $K(g^S) = S$. Then $W(g^S) > W(g^T)$ when $v < \beta$.$\square$

Before proving Proposition 5 we introduce Lemma A3:

Lemma A3. *Suppose $g' = g + ij$, $i \in K(g)$, $j \notin K(g)$ and $K(g+ij) = K(g)$, then $W(g') - W(g) > 0$ if $v_i(g) \geq v_k(g)$ for all $k \in N$.*

Proof of Lemma A3.

Let $g' = g + ij$, $i \in K(g)$, $j \notin K(g)$ and $K(g + ij) = K(g)$. We then have that

$$W(g') - W(g) = \beta + \frac{(\kappa(g))^2 - 1}{\kappa(g)} \left(h_{\kappa(g)}(g) - h_{\kappa(g)}(g + ij) \right)$$

It follows that

$$W(g') - W(g) = \beta + \left(\kappa(g)^2 - 1 \right) \left(-\frac{\frac{\beta}{(v_i(g) + \beta)v_i(g)}}{\sum_{k \in K(g)} \frac{1}{v_k(g)} \sum_{k \in K(g)} \frac{1}{v_k(g + ij)}} \right)$$

Rewriting we obtain:

$$W(g') - W(g) = \beta \left(1 - \frac{(\kappa(g))^2 - 1}{\sum_{k \in K(g)} \frac{v_i(g) + \beta}{v_k(g + ij)} \sum_{k \in K(g)} \frac{v_i(g)}{v_k(g)}} \right).$$

Using $v_i(g) \geq v_k(g)$ for all $k \in N$, we have that $\sum_{k \in K(g)} \frac{v_i(g)}{v_k(g)} \geq \kappa(g)$ and $\sum_{k \in K(g)} \frac{v_i(g) + \beta}{v_k(g + ij)} \geq \kappa(g)$. It follows that $W(g') - W(g) \geq \beta \left(1 - \frac{(\kappa(g))^2 - 1}{(\kappa(g))^2} \right) > 0$.□

Proof of Proposition 5.

Suppose first that $v/\beta \leq 1$. Then, simple calculations show that $W(g^*) > W(g^N) > W(g^S)$ for all pairwise equilibrium g^S, where g^* is the star network.[15] If, on the other hand, $v/\beta > 1$, then, $W(g^S + ij) > W(g^S) > W(g^N)$ for $i \in S$ and $j \in N \backslash S$. Indeed, $j \notin K(g^S + ij)$ as otherwise g^S would not be a pairwise equilibrium. In addition, $k \in K(g^S + ij)$ for $k \in S \backslash \{i\}$ since the participation constraint of the agent k such that $\phi(k, g) = 2$ is always satisfied. Thus $K(g^S + ij) = S$ and Lemma A3 applies.□

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[15]Every agent participates both in the complete and in the star network. We have $W(g^*) > W(g^N) \iff v/\beta < n^2 - 3n + 1$.