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Paul Belleflamme and Valeria Forlin

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Endogenous vertical segmentation in a Cournot oligopoly*

Paul Belleflamme[†]
Université catholique de Louvain

Valeria Forlin[‡]
Université catholique de Louvain

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Abstract

An arbitrary number of (ex ante symmetric) firms first choose whether to produce a high-quality or a low-quality product and then, the quantity of product to put on the market. We establish the following results: (i) there exists competition within and across quality segments; (ii) firms may be better off producing the low quality if competition within this segment is sufficiently low; (iii) a firm's switch across qualities may benefit all the other firms; (iv) there exists a unique partition of the firms between the two quality segments; (v) if high quality has a larger cost-quality ratio, then the equilibrium exhibits vertical differentiation; (vi) there may be too much differentiation from the consumers' point of view.

Keywords: Quality, differentiation, oligopolistic competition
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[†]Université catholique de Louvain, CORE and LIDAM, Voie du Roman Pays 34, 1348 Louvain-la-Neuve (Belgium); Paul.Belleflamme@uclouvain.be. Other affiliation: CESifo.

[‡]Université catholique de Louvain, CORE and LIDAM, and European Commission, DG CLIMA, Avenue de Beaulieu 24, 1160 Brussels (Belgium); valeria.forlin@ec.europa.eu.

1 Introduction

Many oligopolistic markets share the following features: consumers can choose between two qualities of products or services (high or low), each quality is produced by a separate set of competing firms, and quality levels are mostly exogenous to the firms. The clearest examples are markets with certified and non-certified products (think, e.g., of MSC certified fish or EU Ecolabel products). Other examples are industries, such as airlines, retail or hotels, where some firms choose to adopt a low cost business model, so as to offer lower-quality alternatives to the products or services proposed by traditional firms.

In these markets, it is important to understand how firms decide which quality to produce. In the above examples, we need to ask what motivates firms to apply for certification or to adopt a low cost business model. Clearly, firms will base their decision on the relative profitability of producing one or the other quality. Yet, it is not clear how to determine this relative profitability, as it depends on both exogenous and endogenous factors. The exogenous factors are the consumers' willingness to pay for quality upgrades and the respective costs of producing the two qualities; the endogenous factors are the decisions of all firms, as they will jointly determine the level of competition that will prevail on each quality segment. This note aims at understanding better the interplay between these factors.

To address this issue, we analyze a two-stage model of product competition: firms commit first to produce either the high or the low quality; next, they set the quantity of product to put on the market. We establish the following results. In terms of quantity competition (second stage), we highlight the existence of an 'own-competition' and a 'cross-competition' effect: entry in one quality segment reduces equilibrium profits not only in that segment but also in the other segment. A second result is that firms may be better off producing the low quality if competition within this segment is sufficiently low. Regarding quality competition (first stage), we first show that when a firm switches from one quality to the other, this moves has ambiguous impacts on the other firms: as expected, it could hurt the firms in the segment that is joined and benefit firms in the segment that is left, but it can also hurt–or benefit–all the other firms. Next, we prove the existence of a unique partition of the firms between the two quality segments. We further show that a sufficient condition for vertical differentiation (i.e., both qualities being chosen at equilibrium) is that the cost-quality ratio be larger for the high than for the low quality. Finally, we illustrate the possibility of disagreement between firms and consumers: firms' decisions may lead to vertical differentiation while consumers would prefer the production of a single quality.

Our analysis contributes to the abundant literature on vertical differentiation. This literature, following the seminal papers of Gabszewicz and Thisse (1979), and Shaked and Sutton (1982), has mostly focused on duopolistic Bertrand competition. Obviously, this setting is ill-suited to represent the industries that we are studying here, as each quality can potentially be produced by several firms. We therefore chose to depart from the usual model by considering Cournot competition among an arbitrary number of firms.[1] To the best of our knowledge, this

[1] An alternative route would have been to stick to price competition and to introduce horizontal differentiation

model has not been solved so far; our characterization of the second-stage equilibrium is thus a novel result. Other papers consider Cournot competition in vertically differentiated markets but adopt different settings and focus on different issues. Gal Or (1983, 1985 and 1987) considers the interplay between quality and quantity oligopolistic competition in similar, yet different, frameworks than ours.[2] Bonanno (1986) examines duopolistic competition with successive quality and quantity decisions; under the assumption that the two qualities are produced at zero cost, he shows that vertical differentiation does not emerge at equilibrium; we generalize this result with more than two firms and potentially different costs for the two qualities. Motta (1993) compares Cournot with Bertrand competition in a duopoly framework, with a continuous choice of quality levels in the first stage. Turrini (2000) considers, like us, a Cournot model of vertical differentiation with two qualities and an arbitrary number of firms, but he adopts the strong assumption that the total output of the high-quality segment is fixed; in our model, we do not impose this restriction. Johnson and Myatt (2006) assume a Cournot industry with N multi-product firms; in a similar setting, with two qualities and a commitment to quality before the choice of quantities, they analyze how profits are affected by the choice of producing only one type of quality or both; however, they do not examine how the number of firms making the same quality choice affects each firm's decision. Miao and Van Long (2017) adopt a similar setting to the one in Johnson and Myatt (2006), but they look for a free-entry equilibrium, and study the implications of an increase in market size.

2 Equilibrium quality and quantity decisions

2.1 The model

A unit mass of consumers are identified by their valuation for quality improvement, θ, which is assumed to be uniformly distributed on the unit interval. A consumer of type θ obtains utility $\theta s_k - p_k$ from one unit of product k that has quality s_k, and is sold at price p_k. Two qualities are available: a high (s_h) and a low (s_l) quality, with $s_h > s_l > 0$. A unit of quality s_k is produced at a constant unit cost c_k, $k = h, l$. The industry is composed of N identical firms. We analyze the following three-stage game. At the first stage, firms simultaneously decide whether to produce the high or the low quality. At the second-stage, firms compete à la Cournot; that is, they choose which quantity to produce given the market-clearing prices. At the third-stage, consumers observe the prices and decide to buy a unit of high quality, a unit of low quality, or nothing. We solve the game for its subgame-perfect equilibrium.

Note that we assume that firms can only choose to produce a single quality at the first stage of the game. This assumption squares well with situations where producing the high quality

in each quality segment (in the spirit of Häckner, 2000).

[2] The first paper considers free entry, more than two qualities and the possibility for firms to produce several qualities; in the second paper, quality and quantity decisions are simultaneous, and quality levels are endogenous; the third paper is closer to ours in that quality choices precede quantity choices but the analysis is restricted to a duopoly.

requires being granted some label or certification, which is incompatible with continuing to produce the low quality. For instance, the certification for organic food may not be granted if the non-organic variety is produced on the same farm, to avoid fraud. Another reason could be that a firm producing a high-quality good (e.g., a luxury good) may fear that its reputation could be tainted if it was known to produce a low-quality (or mass-market) good as well; concurrently, a low-quality producer (e.g., a hard discounter) may not be perceived as a credible producer of high-quality goods.[3]

To solve the third-stage of the game, identify as θ_{hl} the consumer who is indifferent between buying either the high or the low quality, and by $\theta_{l\emptyset}$, the consumer who is indifferent between buying the low quality or nothing. We easily find that $\theta_{hl} = (p_h - p_l)/(s_h - s_l)$ and $\theta_{l\emptyset} = p_l/s_l$. As long as $p_h/s_h > p_l/s_l$, consumers with $\theta_{hl} \leq \theta \leq 1$ buy a unit of the high quality, consumers with $\theta_{l\emptyset} \leq \theta \leq \theta_{hl}$ buy a unit of the low quality, and consumers with $0 \leq \theta \leq \theta_{l\emptyset}$ buy nothing. The demand functions for the two goods are therefore $Q_h = 1 - \theta_{hl}$ and $Q_l = \theta_{hl} - \theta_{l\emptyset}$. Inverting this system, we get the following system inverse demand functions:

$$\begin{cases} p_h = s_h(1 - Q_h - \frac{s_l}{s_h}Q_l), \\ p_l = s_l\left(1 - Q_h - Q_l\right). \end{cases}$$

Moving backwards, we now characterize the equilibrium of the second stage (quantity choices) and first stage (quality choices) of the game.

2.2 Quantity equilibrium

Suppose that at the first-stage, n_h (resp. n_l) firms have decided to produce the high (resp. low) quality, with $n_h + n_l = N$. A typical firm i producing the high quality chooses its quantity $q_{h,i}$ to maximize $s_h(1 - (q_{h,i} + q_{h,-i}) - \frac{s_l}{s_h}Q_l)q_{h,i} - c_h q_{h,i}$, where $q_{h,-i}$ denotes the total quantity of the high-quality product produced by the other $(n_h - 1)$ firms. Similarly, a typical firm j producing the low quality chooses its quantity $q_{l,j}$ to maximize $s_l\left(1 - Q_h - (q_{l,j} + q_{l,-j})\right)q_{l,j} - c_l q_{l,j}$. The first-order conditions give, respectively, $2s_h q_{h,i} = s_h - c_h - s_h q_{h,-i} - s_l Q_l$ and $2s_l q_{l,j} = s_l - c_l - s_l Q_h - s_l q_{l,-j}$. As firms are ex ante symmetric, we have that at equilibrium, each of the n_h (resp. n_l) firms produce the same quantity q_h (resp. q_l) of the high- (resp. low-) quality product. The previous equations can thus be rewritten as $2q_h s_h = s_h - c_h - s_h\left(n_h - 1\right)q_h - s_l n_l q_l$ and $2q_l s_l = s_l - c_l - s_l n_h q_h - s_l\left(n_l - 1\right)q_l$. To ease the exposition, we define $v_k \equiv s_k - c_k$ $(k = h, l)$, which can be interpreted as the social value of a unit of quality k (i.e., the difference between the highest willingness to pay – θs_k evaluated at $\theta = 1$ – and the unit cost of production). We

[3] For instance, Renault uses the brand Dacia to market cheaper and lower quality cars; BNP-Paribas launched its online bank under a separate brand name (Hello bank!). In the other direction, commentators raised eyebrows when Lidl (a hard discounter) announced a collaboration with top-model Heidi Klum to distribute 'high-end yet affordable' fashion (see, `https://bit.ly/2Sd7rDR`).

can then express the equilibrium Cournot quantities as

$$\begin{aligned} q_h(n_h, n_l) &= \frac{(n_l+1)\, v_h - n_l v_l}{s_h(n_h+1)(n_l+1) - s_l n_h n_l}, \\ q_l(n_h, n_l) &= \frac{\frac{s_h}{s_l}(n_h+1)\, v_l - n_h v_h}{s_h(n_h+1)(n_l+1) - s_l n_h n_l}. \end{aligned}$$

To guarantee positive equilibrium quantities in any subgame, we impose:

$$\text{(A)}\ \frac{N-1}{N} < \frac{v_h}{v_l} < \frac{N}{N-1}\frac{s_h}{s_l}.$$

Using the first-order conditions, we find that at equilibrium, $s_k q_k = p_k - c_k$. It follows that equilibrium profits are computed as $\pi_k = s_k q_k^2$, $k = h, l$. That is,

$$\pi_h(n_h, n_l) = s_h \left(\frac{(n_l+1)\, v_h - n_l v_l}{s_h(n_h+1)(n_l+1) - s_l n_h n_l} \right)^2, \tag{1}$$

$$\pi_l(n_h, n_l) = s_l \left(\frac{\frac{s_h}{s_l}(n_h+1)\, v_l - n_h v_h}{s_h(n_h+1)(n_l+1) - s_l n_h n_l} \right)^2. \tag{2}$$

In the corner cases where all firms produce the same quality, we have

$$\pi_h(N, 0) = \frac{v_h^2}{s_h(N+1)^2}, \text{ and } \pi_l(0, N) = \frac{v_l^2}{s_l(N+1)^2}.$$

Before turning to the first stage, we stress two important results. First, the arrival of an additional firm, whichever the quality it produces, reduces the equilibrium profits of any firm (see Appendix 4.1 for the proof):

$$\begin{aligned} \pi_h(n_h, n_l) > \pi_h(n_h+1, n_l) \text{ and } \pi_h(n_h, n_l) > \pi_h(n_h, n_l+1), \\ \pi_l(n_h, n_l) > \pi_l(n_h+1, n_l) \text{ and } \pi_l(n_h, n_l) > \pi_l(n_h, n_l+1). \end{aligned}$$

That is, there is an 'own-competition' effect (the top line) and also a '*cross*-competition' effect (the bottom line). Second, the relative profitability of producing one or the other quality depends not only on the exogenous parameters (s_k and c_k, $k \in \{h, l\}$) but also on the endogenous split of the firms between the two qualities. Comparing equilibrium profits in the two segments, we find

$$\pi_h(n_h, n_l) > \pi_l(n_h, n_l) \iff \frac{v_h}{v_l} > \sigma + \sigma \frac{(\sigma-1)(n_h - n_l)}{\sigma + n_h + \sigma n_l}.$$

We see that if the same number of firms compete in the two segments ($n_h = n_l$), $v_h > v_l$ (or $s_h - c_h > s_l - c_l$) is not a sufficient condition for the high quality to be more profitable than the low quality: it is necessary that $v_h/v_l > \sigma$, where $\sigma \equiv \sqrt{s_h/s_l} > 1$. We also see that if there is more competition in the high- than in the low-quality segment ($n_h > n_l$), then the ratio v_h/v_l must be even larger for profits to be higher in the high-quality segment. There is thus a trade-off between the social value of a market segment (v_h vs. v_l) and the intensity of competition within this segment (n_h vs. n_l).

2.3 Quality equilibrium

In the first stage, firms choose which quality to produce, anticipating the equilibrium of the ensuing Cournot competition. This game can be seen as a coalition game with simultaneous decisions and open membership. By choosing to produce either the high or the low quality, firms determine a 'coalition structure' (i.e., a partition of the set of firms into disjoint coalitions), with each firm's profit being a function of the whole coalition structure.[4] Before we characterize the equilibrium partition structure, we want to analyze the externalities among firms.

Externalities. Our analysis of the second-stage equilibrium taught us that a firm's equilibrium profit decreases when an additional firm enters its own coalition or the other coalition, other things being equal. At the first stage of the game, however, other things are not equal, insofar as an additional firm in one coalition necessarily means one less firm in the other coalition (recall that $N = n_h + n_l$ is fixed). Therefore, the impact of entry into a coalition depends on the balance between two conflicting forces: by switching, a firm increases competition in the coalition that it joins (a negative 'own-competition' effect) but also reduces competition in the coalition that it leaves (a positive 'cross-competition' effect). The net effect is a priori ambiguous, as it depends on which of the two externalities is stronger. The same two externalities also condition the evolution of the profits of the firms in the coalition left by the switching firm.

To see this more formally, suppose that one firm switches from the low-quality to the high-quality coalition. The change in profits in the two coalitions can be decomposed as follows (with $k \in \{h, l\}$):

$$\begin{aligned}\pi_k(n_h+1, n_l-1) - \pi_k(n_h, n_l) &= \underbrace{\pi_k(n_h+1, n_l-1) - \pi_k(n_h, n_l-1)}_{-} \\ &\quad + \underbrace{\pi_k(n_h, n_l-1) - \pi_k(n_h, n_l)}_{+}.\end{aligned}$$

We show in Appendix 4.1 that

$$\begin{aligned}\pi_h(n_h+1, n_l-1) &> \pi_h(n_h, n_l) \Leftrightarrow \frac{v_h}{v_l} < v_1(n_h), \\ \pi_l(n_h+1, n_l-1) &> \pi_l(n_h, n_l) \Leftrightarrow \frac{v_h}{v_l} < v_2(n_h), \\ \text{with } \frac{N-1}{N} &< v_1(n_h) < v_2(n_h) < \frac{N}{N-1}\frac{s_h}{s_l}.\end{aligned}$$

For a given value of n_h (and $n_l = N - n_h$), there are thus three configurations of externalities. If $v_1(n_h) < v_h/v_l < v_2(n_h)$, we have that when one firm switches from the low quality to the high quality, high-quality firms are worse off and low-quality firms are better off. This result is in line with what we expect, as competition increases in the high-quality coalition and decreases in the low-quality coalition. However, the switch could also be detrimental for all firms (i.e., externalities are negative for both coalitions); this is so if $v_h/v_l > v_2(n_h)$. Finally,

[4] For a survey of approaches to coalition formation, see Bloch (2002).

the switch could benefit all firms (i.e., externalities are *positive* for both coalitions); this is so if $v_h/v_l < v_1(n_h)$. In the latter case, because the low quality has a relatively high social value (compared to the high quality), high-quality firms perceive low-quality firms as "tough" competitors, so much so that one fewer competitor from the low-quality coalition is beneficial to their profits, even if this means a stronger level of competition in their own coalition; that is, the positive 'cross-competition' effect outweighs the negative 'own-competition' effect; by the same token, remaining firms in the low-quality competition also benefit from the switch. We show furthermore that both v_1 and v_2 increase in n_h (given $n_l = N - n_h$); this means that, high-quality (resp. low-quality) firms are more likely to welcome entry in their coalition when n_h is large (resp. small). Again, this is explained by the fact that, in these cases, competitors from the other coalition are perceived as tougher competitors, because they face weaker competition within their coalition.

Equilibrium. To have an interior equilibrium with n_h firms producing the high quality and $n_l = N - n_h$ firms producing the low quality, with $1 \leq n_h \leq N - 1$, the following two conditions must be satisfied: $\pi_h(n_h, n_l) \geq \pi_l(n_h - 1, n_l + 1)$ and $\pi_l(n_h, n_l) \geq \pi_h(n_h + 1, n_l - 1)$; the former (resp. latter) condition states that no firm producing the high quality (resp. the low quality) wants to switch to the other quality. To facilitate the exposition, we rewrite the two conditions as

$$U(n_h) \leq 0 \leq U(n_h - 1),$$

where $U(n_h) \equiv \pi_h(n_h + 1, n_l - 1) - \pi_l(n_h, n_l)$ measures a firm's '*upgrade incentive*' (i.e., the incentive to move from the low to the high quality) when n_h firms produce the high quality. Using this notation, we express the condition for a corner equilibrium where all firms produce the high quality as $U(N - 1) \geq 0$, and the condition for a corner equilibrium where all firms produce the *low* quality as $U(0) \leq 0$.

We are now in a position to state our main results.

Proposition 1 *(1) The equilibrium of the quality game exists and is unique. (2). As the ratio of the social values, v_h/v_l, increases, more firms produce the high quality at equilibrium. (3) The equilibrium involves vertical differentiation (i.e., both qualities are produced) unless the social values of the two qualities are either very close or very distant from one another.*

We just sketch the proof here (the details can be found in Appendix 4.2). Existence and unicity follow from the fact that the conditions $U(n_h) \leq 0 \leq U(n_h - 1)$ can be rewritten as $V(n_h - 1) \leq v_h/v_l \leq V(n_h)$, and that $V(\cdot)$ is a decreasing function. So, as n_h goes from 0 to $N - 1$, the function $V(\cdot)$ partitions the real line in N intervals, meaning that the ratio v_h/v_l necessarily falls in one and only one of these intervals. The second statement follows directly: larger values of the ratio v_h/v_l go above higher values of $V(\cdot)$, which correspond to equilibria with more firms choosing to produce the high quality (which is quite intuitive as its relative social value increases). Finally, vertical differentiation prevails if the equilibrium is interior, which requires that $U(N-1) < 0 < U(0)$, or equivalently $V(0) < v_h/v_l < V(N-1)$. Simple computations establish that

$$U(N-1) < 0 < U(0) \iff \Phi \frac{s_h}{s_l} > \frac{v_h}{v_l} > \Phi^{-1}$$

$$\text{with } \Phi \equiv \frac{N(N+1)\sigma}{2N\sigma^2 + (N-1)[(N+1)\sigma - 1]} \text{ and } \sigma \equiv \sqrt{s_h/s_l}.$$

It is easily shown that $\Phi < N/(N-1)$, which implies that

$$\frac{N-1}{N} < \Phi^{-1} < \Phi \frac{s_h}{s_l} < \frac{N}{N-1} \frac{s_h}{s_l}.$$

As a consequence, corner equilibria remain possible in the space of parameters defined by Assumption (A): all firms adopt the low quality for $(N-1)/N < v_h/v_l < \Phi^{-1}$ and all firms adopt the high quality for $\Phi < v_h s_l / v_l s_h < N/(N-1)$.

We saw in the previous section that if $v_h/v_l > \sigma$, then high quality is 'more profitable' than low quality (in the sense that if competition has the same intensity in the two segments, firms make larger profits when they produce the high quality). Unsurprisingly, if this is so, some firms will choose to produce the high quality at equilibrium. It is indeed easy to show that $\sigma > \Phi^{-1}$, meaning that $v_h/v_l > \sigma$ implies that $U(0) > 0$, or $\pi_h(1, N-1) > \pi_l(0, N)$.

We can also derive a necessary condition on costs and qualities for vertical differentiation to take place at equilibrium. For at least one firm to adopt the low quality at equilibrium, we need $\pi_l(N-1, 1) > \pi_h(N, 0)$, or $U(N-1) < 0$, or $\frac{v_h}{v_l} < \Phi \frac{s_h}{s_l}$. Using the definition of Φ and recalling that $\sigma > 1$, it is readily shown that $\Phi < 1$. So, to have $U(N-1) < 0$, it must be that

$$\frac{v_h}{v_l} < \frac{s_h}{s_l} \Leftrightarrow \frac{c_h}{s_h} > \frac{c_l}{s_l}. \tag{3}$$

We record this result in the following corollary:

Corollary 1 *For both qualities to be produced at equilibrium (i.e., for vertical differentiation to prevail), a necessary condition is that the cost-quality ratio be larger for the high than for the low quality:* $c_h/s_h > c_l/s_l$.

To illustrate the scope of the latter condition, let us examine the two canonical cases that are considered in the literature on vertical differentiation (see, e.g., Motta, 1993). In the first case, it is assumed that there are only fixed costs of quality improvement (while variable costs do not change with quality). That would mean here that $c_h = c_l = c$. Given that $s_h > s_l$, condition (3) would then be violated, meaning that vertical differentiation would not endogenously emerge (in fact, all firms would adopt the high quality at equilibrium). The second case takes the opposite assumption: no fixed cost but a variable cost that increases with quality. In particular, it is usually posited that the marginal cost of production of one unit of quality k is proportional to the square of the quality level: $c_k = \alpha s_k^2$. Then, condition (3) is equivalent to $s_h > s_l$, which is satisfied by definition; in this case, vertical differentiation may emerge at equilibrium.

3 Concluding remarks

To complete the analysis, we compare quality choices under Cournot and Bertrand competition, and we derive some results about consumer surplus.

Comparison with price competition. Consider first stage 2 under price competition. If several firms produce the same quality (i.e., if $n_k > 1$, $k = h, l$), then their equilibrium profits go to zero (i.e., the unique Nash equilibrium for these firms is to set $p_k = c_k$, as quality k is a homogeneous product).[5] So, the only case of interest is the one studied in the literature with two firms, each producing a different quality (i.e., $n_h = n_l = 1$). In this case, equilibrium profits are computed as (with the superscript b standing for Bertrand competition):

$$\pi_h^b = \frac{\left((2s_h - s_l)\, v_h - s_h v_l\right)^2}{(s_h - s_l)\,(4s_h - s_l)^2}, \text{ and } \pi_l^b = \frac{s_h}{s_l}\frac{\left((2s_h - s_l)\, v_l - s_l v_h\right)^2}{(s_h - s_l)\,(4s_h - s_l)^2}, \tag{4}$$

where it must be assumed that

$$\frac{s_h}{2s_h - s_l} < \frac{v_h}{v_l} < \frac{2s_h - s_l}{s_l}. \tag{5}$$

Note that the latter set of conditions is more restrictive than assumption (A) for $N = 2$. We show in Appendix 4.3 that $\pi_h^c(1,1) - \pi_h^b = \pi_l^c(1,1) - \pi_l^b > 0$, which implies that *the market is more competitive when firms sets prices rather than quantities*.[6]

The real difference between the two models concerns the sensitivity of profits to changes in quality levels (assuming $c_h \geq c_l$). In Appendix 4.4, we show that the results are qualitatively similar in the two models for the high-quality firm: both equilibrium profits increase with s_h and decrease with s_l. However, results are contrasted for the low-quality firm. First, the equilibrium profit increases with s_l under Cournot competition but decreases with s_l under Bertrand competition. Second, the equilibrium profit decreases with s_h under Cournot competition but may increase with s_h under Bertrand competition (if v_h/v_l is large enough). Hence, *the low-quality firm may welcome an increase in the quality difference in the Bertrand game, but it never does so in the Cournot game* (as its profits are reduced when either s_h increases or s_l decreases).

Consumer surplus. The consumer surplus for a given partition of the N firms between the two segments is computed as:

$$CS\,(n_h, n_l) = \int_{\theta_{hl}}^{1} (\theta s_h - p_h)\, d\theta + \int_{\theta_{l\emptyset}}^{\theta_{hl}} (\theta s_l - p_l)\, d\theta = \tfrac{1}{2}\left(Q_h^2 s_h + Q_l^2 s_l + 2Q_h Q_l s_l\right),$$

[5] At stage 1, the quality game has multiple Nash equilibria: any situation where a single firm produces either the high or the low quality and all other firms produce the other quality is a Nash equilibrium (the single firm makes positive profits and would make zero profits if it deviated; all the other firms are indifferent, as they make zero profits whichever quality they decide to produce). It follows that there is always product differentiation at equilibrium in the Bertrand game: if all firms were producing the same quality, any firm would find it profitable to deviate so as to be the sole producer of the other quality and, thereby, achieve positive profits.

[6] This generalizes the result in Motta (1993) where $c_h = c_l = 0$.

where the second equality uses the definitions of p_h and p_l, as well as the facts that $\theta_{hl}(n_h, n_l) = 1 - Q_h(n_h, n_l)$ and $\theta_{l\emptyset} = 1 - Q_h(n_h, n_l) - Q_l(n_h, n_l)$. When all firms produce the high quality, $CS(N, 0) = \frac{1}{2}s_h Q_h(N, 0)^2 = \frac{1}{2}N^2\pi_h(N, 0)$. Similarly, when all firms produce the low quality, $CS(0, N) = \frac{1}{2}N^2\pi_l(0, N)$. So, if $v_h/v_l > \sigma \equiv \sqrt{s_h/s_l}$, consumers (like firms) prefer a situation where all firms produce the high quality to a situation where they all produce the low quality. The latter condition does not imply, however, that consumer surplus is maximized when all firms produce the high quality; they may indeed be better off if some firms produce the low quality instead, as evidenced by the following example. Take $N = 10$, $s_h = 4$ and $c_h = 1$ (so that $v_h = 3$), $s_l = 1$ and $c_l = 0$ (so that $v_l = 1$); note that $\sigma \equiv \sqrt{s_h/s_l} = 2 > 1$; note also that Assumption (A) is satisfied as $9/10 < v_h/v_l = 3 < 40/9$. We compute $CS(10, 0) = 0.930 < CS(9, 1) = 0.936$. In this case, when going from 10 to 9 firms producing the high quality, the decrease in Q_h is more than compensated by the increase in Q_l.

As the following example illustrates, *the equilibrium may exhibit too much vertical differentiation from the consumers' viewpoint.* Take $N = 2$ and $\sigma = 2$. Then, the equilibrium is such that both firms adopt the low quality if $v \equiv v_h/v_l \leq 1.75$, both firms adopt the high quality if $v \geq 2.29$ and firms split between the two qualities if $1.75 \leq v \leq 2.29$. As for consumers, they prefer that both firms adopt the low quality if $v \leq 1.85$, both firms adopt the high quality if $v \geq 2.17$, and firms split for $1.85 \leq v \leq 2.17$. Hence, there is too much differentiation for $1.75 \leq v \leq 1.85$ and for $2.17 \leq v \leq 2.29$: in both cases, firms split while consumers would prefer that they both produce the low quality (in the former case) or the high quality (in the latter case).

4 Appendix

4.1 Own- and cross-competition effect

We first establish that equilibrium profits decrease in the number of firms in any segment. We have:

$$\begin{aligned}\pi_h(n_h, n_l) &> \pi_h(n_h+1, n_l+1) \Leftrightarrow \tfrac{(n_l+1)v_h - n_l v_l}{s_h(n_h+1)(n_l+1) - s_l n_h n_l} > \tfrac{(n_l+1)v_h - n_l v_l}{s_h(n_h+2)(n_l+1) - s_l(n_h+1)n_l} \\ &\Leftrightarrow \frac{v_h}{v_l} > \frac{n_l}{n_l+1}, \text{ which follows from Assumption (A);}\end{aligned}$$

$$\begin{aligned}\pi_h(n_h, n_l) &> \pi_h(n_h, n_l+1) \Leftrightarrow \tfrac{(n_l+1)v_h - n_l v_l}{s_h(n_h+1)(n_l+1) - s_l n_h n_l} > \tfrac{(n_l+2)v_h - (n_l+1)v_l}{s_h(n_h+1)(n_l+2) - s_l n_h(n_l+1)} \\ &\Leftrightarrow \frac{v_h}{v_l} < \frac{n_h+1}{n_h}\frac{s_h}{s_l}, \text{ which follows from Assumption (A);}\end{aligned}$$

$$\begin{aligned}\pi_l(n_h, n_l) &> \pi_l(n_h+1, n_l) \Leftrightarrow \tfrac{s_h(n_h+1)v_l - s_l n_h v_h}{s_h s_l(n_h+1)(n_l+1) - s_l^2 n_h n_l} > \tfrac{s_h(n_h+2)v_l - s_l(n_h+1)v_h}{s_h s_l(n_h+2)(n_l+1) - s_l^2(n_h+1)n_l} \\ &\Leftrightarrow \frac{v_h}{v_l} > \frac{n_l}{n_l+1}, \text{ which follows from Assumption (A);}\end{aligned}$$

$$\pi_l(n_h,n_l) > \pi_l(n_h,n_l+1) \Leftrightarrow \frac{s_h(n_h+1)v_l - s_l n_h v_h}{s_h s_l (n_h+1)(n_l+1) - s_l^2 n_h n_l} > \frac{s_h(n_h+1)v_l - s_l n_h v_h}{s_h s_l (n_h+1)(n_l+2) - s_l^2 n_h (n_l+1)}$$

$$\Leftrightarrow \frac{v_h}{v_l} < \frac{n_h+1}{n_h}\frac{s_h}{s_l}, \text{ which follows from Assumption (A).}$$

Next, we establish the results regarding the externalities among firms in the first stage of the game. As for the equilibrium profits of firms producing the high quality, we have:

$$\pi_h(n_h+1,n_l-1) > \pi_h(n_h,n_l) \Leftrightarrow \frac{n_l v_h - (n_l-1)v_l}{s_h(n_h+2)n_l - s_l(n_h+1)(n_l-1)} > \frac{(n_l+1)v_h - n_l v_l}{s_h(n_h+1)(n_l+1) - s_l n_h n_l}$$

$$\Leftrightarrow \frac{v_h}{v_l} < \frac{s_h(n_h+1) + n_l s_l + n_l^2(s_h-s_l)}{s_l(n_h+1) + n_l s_h + n_l^2(s_h-s_l)} \equiv v_1(n_h).$$

In the low-quality segment, we have:

$$\pi_l(n_h+1,n_l-1) > \pi_l(n_h,n_l) \Leftrightarrow \frac{\frac{s_h}{s_l}(n_h+2)v_l - (n_h+1)v_h}{s_h(n_h+2)n_l - s_l(n_h+1)(n_l-1)} > \frac{\frac{s_h}{s_l}(n_h+1)v_l - n_h v_h}{s_h(n_h+1)(n_l+1) - s_l n_h n_l}$$

$$\Leftrightarrow \frac{v_h}{v_l} < \frac{s_h}{s_l}\frac{s_h(n_h+1) + s_l n_l + (n_h+1)^2(s_h-s_l)}{s_l(n_h+1) + s_h n_l + (n_h+1)^2(s_h-s_l)} \equiv v_2(n_h).$$

We compute

$$v_2(n_h) - v_1(n_h) = \frac{(s_h-s_l)((s_l+n_l(s_h-s_l))n_h + (s_l+n_l(2s_h-s_l)))((s_h+n_l(s_h-s_l))n_h + s_h(n_l+1))}{s_l(s_l(n_h+1)+s_h n_l+(n_h+1)^2(s_h-s_l))\left(s_l(n_h+1)+n_l s_h+n_l^2(s_h-s_l)\right)} > 0.$$

We now show that both $v_1(n_h)$ and $v_2(n_h)$ increase in n_h. Replacing n_l by $N-n_h$, we have:

$$\frac{\partial v_1}{\partial n_h} = (s_h-s_l)\frac{[(s_h-s_l)(N+2n_h(N-n_h-1)) + s_h(2N+1) + s_l]}{[(s_h-s_l)((N-n_h)^2-n_h) + s_h N + s_l]^2},$$

$$\frac{\partial v_2}{\partial n_h} = \frac{s_h(s_h-s_l)}{s_l}\frac{[2n_h(s_h-s_l)(N-n_h-1) + (2s_h(N+1) + (N-1)(s_h-s_l))]}{[(s_h-s_l)n_h(n_h+1) + s_h(N+1)]^2},$$

both of which are clearly positive.

As a result, $v_1(n_h)$ reaches its smallest value at $n_h=0$ and $n_l=N$, while $v_2(n_h)$ reaches its largest value at $n_h=N-1$ and $n_l=1$; we compute:

$$v_1(0) - \frac{N-1}{N} = \frac{s_l + (2s_h-s_l)N}{N(N+1)(s_l+N(s_h-s_l))} > 0,$$

$$\frac{s_h N}{s_l(N-1)} - v_2(N-1) = \frac{s_h}{s_l}\frac{s_l + (2s_h-s_l)N}{(N-1)(s_h+Ns_l+N^2(s_h-s_l))} > 0,$$

which completes the proof (as we consider any switch from the low-quality to the high-quality coalition, which supposes $1 \le n_l \le N$).

4.2 Proof of Proposition 1

Using the definition of $U(n_h)$, we have that $U(n_h) \le 0$ if and only if $\pi_h(n_h+1,n_l-1) \le \pi_l(n_h,n_l)$, which is equivalent to

$$s_h\left(\frac{n_l v_h - (n_l-1)v_l}{s_h(n_h+2)n_l - s_l(n_h+1)(n_l-1)}\right)^2 \le s_l\left(\frac{\frac{s_h}{s_l}(n_h+1)v_l - n_h v_h}{s_h(n_h+1)(n_l+1) - s_l n_h n_l}\right)^2.$$

Under Assumption (A), all equilibrium quantities are positive; we can thus take the square root on both sides of the inequality. Defining $\sigma \equiv \sqrt{s_h/s_l}$, we can then rewrite the previous inequality as

$$\frac{v_h}{v_l} \leq V(n_h) = \frac{V_{num}(n_h)}{V_{den}(n_h)}, \text{ with}$$

$$\begin{aligned} V_{num}(n_h) &\equiv \sigma(n_l-1)\left(\sigma^2(n_h+1)(n_l+1)-n_hn_l\right) \\ &\quad +\sigma^2(n_h+1)\left(\sigma^2(n_h+2)n_l-(n_h+1)(n_l-1)\right), \\ V_{den}(n_h) &\equiv \sigma n_l\left(\sigma^2(n_h+1)(n_l+1)-n_hn_l\right) \\ &\quad +n_h\left(\sigma^2(n_h+2)n_l-(n_h+1)(n_l-1)\right). \end{aligned}$$

Hence, $U(n_h) \leq 0$ if and only if $v_h/v_l \leq V(n_h)$. By analogy, $U(n_h-1) \geq 0$ if and only if $v_h/v_l \geq V(n_h-1)$. To establish the existence and unicity of equilibrium, we need thus to prove that $V(n_h) > V(n_h-1)$, meaning that $V(\cdot)$ is a decreasing function. It is clear that both $V_{num}(n_h)$ and $V_{den}(n_h)$ are positive. The sign of $V(n_h)-V(n_h-1)$ is thus given by the sign of

$$\Delta \equiv V_{num}(n_h)V_{den}(n_h-1) - V_{num}(n_h-1)V_{den}(n_h).$$

A few lines of computations establish that (after substituting $N-n_h$ for n_l)

$$\begin{aligned} \Delta &= \sigma(\sigma-1)\left[\sigma^2+N\sigma^2+n_h\left(\sigma^2-1\right)(N-n_h)\right] \\ &\quad \times\left[\left(\sigma^2-1\right)(N+3\sigma+N\sigma+1)n_h(N-n_h)+\left(N^2\sigma+N^2-2N\sigma^2+\sigma-1\right)\right]. \end{aligned}$$

The first three terms on the top line are clearly positive. As for the fourth term on the second line, it is easily seen that it reaches a maximum at $n_h = N/2$. The smallest value is thus obtained for either $n_h = 1$ or $n_h = N-1$. Evaluating the fourth term at $n_h = 1$ or at $n_h = N-1$, we get

$$\sigma\left((N+3)(N-1)\sigma^2+\left(N^2-2N-1\right)\sigma-2(N-2)\right).$$

This expression is increasing in σ (its derivative with respect to σ is $2(N\sigma-1)(N-2)+\sigma^2(N-1)(N+3)+2\sigma[\sigma(N-1)(N+3)-1]$, all terms of which are positive). As, by definition, $\sigma > 1$, this expression is larger than $(N+3)(N-1)+\left(N^2-2N-1\right)-2(N-2) = 2N(N-1) > 0$. We have thus proved that $V(n_h+1) > V(n_h)$.

To complete the proof, let us show that $V(0) > (N-1)/N$ and $V(N-1) < (s_h/s_l)[N/(N-1)]$. We have

$$\begin{aligned} V(0) &= \frac{2N\sigma^2+(N-1)[(N+1)\sigma-1]}{N(N+1)\sigma} \equiv \Phi^{-1} \text{ and} \\ V(N-1) &= \frac{s_h}{s_l}\frac{N(N+1)\sigma}{2N\sigma^2+(N-1)[(N+1)\sigma-1]} \equiv \frac{s_h}{s_l}\Phi. \end{aligned}$$

It follows that

$$\begin{aligned} V(0)-\frac{N-1}{N} &= \frac{N\left(2\sigma^2-1\right)+1}{N\sigma(N+1)} > 0, \\ \frac{N}{N-1}\sigma^2-V(N-1) &= \frac{N\left(N\left(2\sigma^2-1\right)+1\right)\sigma^2}{(N-1)\left(N\left(2\sigma^2-1\right)+\left(N^2-1\right)\sigma+1\right)} > 0. \end{aligned}$$

4.3 Comparison between price and quantity competition

Using $v = v_h/v_l$ and $s = s_h/s_l$, we can rewrite conditions (5) as

$$\frac{s}{2s-1} < v < 2s - 1.$$

From the first inequality, we deduce that v must be larger than 1/2 (as $s/(2s-1) > 1/2$). We can also reformulate the second inequality as $s > (v+1)/2$.

Developing the profit differences between the Cournot and the Bertrand case, we find:

$$\pi_h^c(1,1) - \pi_h^b = \pi_l^c(1,1) - \pi_l^b = \frac{v_l^2[s(2v-1) - v^2]}{s_l(s-1)(4s-1)^2}.$$

As $s > 1$ and $v > 1/2$, we have that $\pi_h^c(1,1) > \pi_h^b$ and $\pi_l^c(1,1) > \pi_l^b$ if and only if $s > v^2/(2v-1)$, which is satisfied as $(v+1)/2 > v^2/(2v-1)$ for $v > 1/2$.

4.4 Comparative statics with respect to quality

We start with Cournot competition and show that $c_h \geq c_l$ implies that

$$\frac{d\pi_h}{ds_h} > 0,\ \frac{d\pi_h}{ds_l} < 0,\ \frac{d\pi_l}{ds_h} < 0, \text{ and } \frac{d\pi_l}{ds_l} > 0.$$

The four derivatives are computed as:

$$\begin{aligned}
\frac{d\pi_h}{ds_h} &= \tfrac{(n_l+1)v_h - n_l v_l}{(s_h(n_h+1)(n_l+1)-s_l n_h n_l)^2}\left(2s_h(n_l+1) - ((n_l+1)v_h - n_l v_l)\tfrac{s_h(n_h+1)(n_l+1)+s_l n_h n_l}{s_h(n_h+1)(n_l+1)-s_l n_h n_l}\right),\\
\frac{d\pi_l}{ds_l} &= \tfrac{\frac{s_h}{s_l}(n_h+1)v_l - n_h v_h}{(s_h(n_h+1)(n_l+1)-s_l n_h n_l)^2}\left(2s_h(n_h+1)\right)\\
&\quad - \tfrac{\frac{s_h}{s_l}(n_h+1)v_l - n_h v_h}{(s_h(n_h+1)(n_l+1)-s_l n_h n_l)^2}\tfrac{n_h s_l(s_h(n_l+1)(n_h+1)+n_h n_l s_l)v_h + s_h(n_h+1)(s_h(n_l+1)(n_h+1)-3n_h n_l s_l)v_l}{s_l(s_h(n_h+1)(n_l+1)-s_l n_h n_l)},\\
\frac{d\pi_h}{ds_l} &= 2n_l s_h \tfrac{((n_l+1)v_h - n_l v_l)}{(s_h(n_h+1)(n_l+1)-s_l n_h n_l)^2}\left(\tfrac{(n_l+1)v_h - n_l v_l}{s_h(n_h+1)(n_l+1)-s_l n_h n_l} n_h - 1\right),\\
\frac{d\pi_l}{ds_h} &= 2n_h s_l \tfrac{\frac{s_h}{s_l}(n_h+1)v_l - n_h v_h}{(s_h(n_h+1)(n_l+1)-s_l n_h n_l)^2}\left((n_h+1)\tfrac{(n_l+1)v_h - n_l v_l}{s_h(n_h+1)(n_l+1)-s_l n_h n_l} - 1\right).
\end{aligned}$$

- We have that $d\pi_h/ds_h > 0$ if and only if

  $$2s_h(n_l+1) - ((n_l+1)v_h - n_l v_l)\tfrac{s_h(n_h+1)(n_l+1)+s_l n_h n_l}{s_h(n_h+1)(n_l+1)-s_l n_h n_l} > 0.$$

  Using $v_h = s_h - c_h$, $v_l = s_l - c_l$ and simplifying, we can rewrite the latter inequality as

  $$\begin{gathered}(s_h(n_l+1)(n_h+1) + n_h n_l s_l)(c_h + n_l(c_h - c_l))\\ + (s_h + n_l s_h - n_l s_l)^2 n_h + s_h(n_l+1)(s_h + n_l s_h + n_l s_l) > 0,\end{gathered}$$

  which is certainly satisfied if $c_h \geq c_l$.

- We have that $d\pi_l/ds_l > 0$ if and only if

  $$2s_h(n_h+1) > \tfrac{n_h s_l(s_h(n_l+1)(n_h+1)+n_h n_l s_l)v_h + s_h(n_h+1)(s_h(n_l+1)(n_h+1)-3n_h n_l s_l)v_l}{s_l(s_h(n_h+1)(n_l+1)-s_l n_h n_l)}$$

Using $v_h = s_h - c_h$, $v_l = s_l - c_l$ and simplifying, we can rewrite the latter inequality as

$$\begin{aligned}&n_h s_l \left(s_h \left(n_l+1\right)\left(n_h+1\right)+n_h n_l s_l\right) c_h \\ &+\left[\left(n_h+1\right)^2\left(n_l+1\right) s_h-3 n_h n_l s_l\left(n_h+1\right)\right] s_h c_l \\ &+\left(s_h\left(n_l+1\right)\left(n_h+1\right)+n_h n_l s_l\right) s_h s_l>0 .\end{aligned}$$

Evaluating the LHS at $c_h = c_l$, one obtains

$$\begin{aligned}&\left[\left(n_h+1\right)^2\left(n_l+1\right) s_h^2-n_h\left(2 n_l-1\right)\left(n_h+1\right) s_h s_l+n_h^2 n_l s_l^2\right] c_l \\ &+\left(s_h\left(n_l+1\right)\left(n_h+1\right)+n_h n_l s_l\right) s_h s_l>0 .\end{aligned}$$

Let us now show that the term that multiplies c_l is positive. It is a quadratic equation in s_h; this equation does not have real roots as $\left(n_h\left(2n_l-1\right)\left(n_h+1\right)s_l\right)^2 - 4\left(n_h+1\right)^2\left(n_l+1\right)n_h^2 n_l s_l^2 = -n_h^2\left(n_h+1\right)^2\left(8n_l-1\right)s_l^2 < 0$; hence the equation is everywhere positive as $n_h^2 n_l s_l^2 > 0$; it follows that $d\pi_l/ds_l > 0$ for $c_h \geq c_l$.

- We have that $d\pi_l/ds_h < 0$ if and only if

$$\left(n_h+1\right) \tfrac{\left(n_l+1\right) v_h-n_l v_l}{s_h\left(n_h+1\right)\left(n_l+1\right)-s_l n_h n_l}<1$$

$$\begin{aligned}&\Leftrightarrow \quad \left(n_h+1\right)\left(n_l+1\right) v_h-n_l v_l\left(n_h+1\right)<\left(n_h+1\right)\left(n_l+1\right) s_h-s_l n_h n_l \\ &\Leftrightarrow \quad n_h n_l\left(s_l-v_l\right)-n_l v_l<\left(n_h+1\right)\left(n_l+1\right)\left(s_h-v_h\right) \\ &\Leftrightarrow \quad n_h n_l c_l<\left(n_h+1\right)\left(n_l+1\right) c_h+n_l v_l,\end{aligned}$$

which is satisfied if $c_h \geq c_l$. As

$$\left(n_h+1\right) \tfrac{\left(n_l+1\right) v_h-n_l v_l}{s_h\left(n_h+1\right)\left(n_l+1\right)-s_l n_h n_l}>\frac{\left(n_l+1\right) v_h-n_l v_l}{s_h\left(n_h+1\right)\left(n_l+1\right)-s_l n_h n_l},$$

$c_h \geq c_l$, also implies that $d\pi_h/ds_l < 0$.

We now turn to Bertrand competition in the case where $n_h = n_l = 1$. We establish the following results:

$$\frac{d\pi_h^b}{ds_h} > 0,\ \frac{d\pi_h^b}{ds_l} < 0,\ \frac{d\pi_l^b}{ds_l} < 0, \text{ and } \frac{d\pi_l^b}{ds_h} \text{ may be positive.}$$

Recall that the parameter space is retricted by conditions (5), which can be rewritten as

$$\frac{s}{2s-1} < \frac{v_h}{v_l} < 2s-1.$$

We compute

$$\frac{d\pi_h^b}{ds_h} = \frac{\left(2s-1\right)v_h - sv_l}{s_l^2\left(s-1\right)^2\left(4s-1\right)^3}\left[\left(10s-8s^2-5\right)v_h + \left(4s^2+s-2\right)v_l + 2s_l\left(s-1\right)\left(2s-1\right)\left(4s-1\right)\right].$$

As the first fraction is positive, the sign is determined by the bracketed term. Given that $s > 1$, we see that this term is decreasing in v_h/v_l. Hence, this term reaches its smallest value for $v_h/v_l = 2s-1$. Computing this value and recalling that $v_l = s_l - c_l$, we find $(4s-1)(s-1)(s_l + (4s-3)c_l)$, which is clearly positive. This implies that the derivative is positive for all admissible parameters.

We compute

$$\frac{d\pi_h^b}{ds_l} = \frac{(2s-1)v_h - sv_l}{s_l^2(s-1)^2(4s-1)^3}\left[(2s(2s-1)+1)v_h - 3s(2s-1)v_l - 2ss_l(4s-1)(s-1)\right].$$

As the first fraction is positive, the sign is determined by the bracketed term. Given that $s > 1$, we see that this term is increasing in v_h/v_l. Hence, this term reaches its largest value for $v_h/v_l = 2s-1$. Computing this value and recalling that $v_l = s_l - c_l$, we find $-(4s-1)(s-1)(s_l + (2s-1)c_l)$, which is clearly negative. This implies that the derivative is negative for all admissible parameters.

We compute

$$\frac{d\pi_l^b}{ds_l} = \frac{s((2s-1)v_l - v_h)}{s_l^2(s-1)^2(4s-1)^3}\left[\left(2-s-4s^2\right)v_h - \left(9s-18s^2+8s^3-2\right)v_l + 2s_l(s-1)(2s-1)(4s-1)\right].$$

As the first fraction is positive, the sign is determined by the bracketed term. Given that $s > 1$, we see that this term is decreasing in v_h. Hence, this term reaches its largest value for $v_h = \frac{s}{2s-1}v_l$. Computing this value and recalling that $v_l = s_l - c_l$, we find $(4s-1)(s-1)/(2s-1) \times \left(\left(4s^2-5s+2\right)c_l + ss_l(4s-3)\right)$, which is clearly positive. This implies that the derivative is positive for all admissible parameters.

Finally, we compute

$$\frac{d\pi_l^b}{ds_h} = \frac{(2s-1)v_h - sv_l}{s_l^2(s-1)^2(4s-1)^3}\left[(2s(2s-1)+1)v_h - 3s(2s-1)v_l - 2ss_l(4s-1)(s-1)\right].$$

As the first fraction is positive, the sign is determined by the bracketed term. Given that $s > 1$, we see that this term is increasing in v_h/v_l. It follows that the derivative is positive if and only if

$$\frac{v_h}{v_l} > \frac{4s^2-2s+1}{8s^2-4s-1} + 2s\frac{s_l}{v_l}\frac{(4s-1)(s-1)}{8s^2-4s-1} \equiv \mathring{v}.$$

It can be checked that $\mathring{v} > s/(2s-1)$ and that $\mathring{v} < 2s-1$ provided that $2c_l < s_l$. Hence, as long as $2c_l < s_l$, the derivative is positive for $\mathring{v} < v_h/v_l < 2s-1$, which completes the proof.

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