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CHOQUET RATIONALITY

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SOCIAL SCIENCE WORKING PAPER 1000

March 1997

Revised: February 1999

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Abstract

We provide a characterization of the consequences of the assumption that a decision maker with a given utility function is *Choquet rational*: She maximizes expected utility, but possibly with respect to non-additive beliefs, so that her preferences are represented by Choquet expected utility (CEU).

The characterization shows that this notion of rationality allows in general to rationalize more choices than it is possible when beliefs have to be additive. More surprisingly, we find that a considerable restriction on the types of beliefs allowed does not change the set of rational actions. We then remark on the relation between the predictions of CEU model, of a similar model (the maxmin expected utility model), and those of subjective expected utility when the risk attitude of the decision maker is not known. We close with an application of the result to the definition of a solution concept (in the spirit of rationalizability) for strategic-form games.

JEL classification numbers: C72, D81

Key words: Revealed Preferences, Rationalizability, Belief Functions, Choquet Integrals

Choquet Rationality*

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Introduction

In this note we characterize the consequences of the assumption that a decision maker (DM) facing a decision problem is *Choquet rational*. That is, assuming that the DM's von Neumann-Morgenstern utility function is known and fixed, we list all the actions that could be taken by her if she behaves according to Choquet expected utility (CEU) model.

The CEU model was developed by Schmeidler [29] to explain a common pattern of behavior, the most striking instance of which is the well-known Ellsberg paradox [9]: The amount and quality of the information a DM has about the relevant events affects her preferences. In particular, DMs prefer to act on events they feel well-informed about, an attitude which has been called *ambiguity aversion*. Schmeidler noticed that ambiguity aversion can be modelled by extending the standard subjective expected utility (SEU) model to allow the DM to have *non-additive* beliefs (called *capacities*). He also provided an axiomatic foundation and a mathematical representation, using a notion of expectation due to Choquet [5] (hence CEU). A few years earlier, Dempster [6] and Shafer [30] developed the statistics of a special class of capacities, called *belief functions*, as a way to capture imprecise probabilities.

An important, but heretofore unanswered, question is what is the empirical content of the assumption that a DM is described by the CEU model. Clearly, it is important to know what CEU does *not* predict, in order to conclude that it is a useful scientific tool. Here we answer the question for the case in which the utility function of the

* A previous version of this paper was circulated with the title "Choquet Rationalizability." We are indebted to Paolo Battigalli, Kim Border, Larry Epstein, John Ledyard, Massimo Marinacci, Igal Milchtaich, Klaus Nehring and audiences at Berkeley, Caltech, Paris VI and the LOFT 2 conference (Turin, December 1996), the Conference on "Rationality in Game Theory and Utility Theory" (Caen, June 1997) and ESEM 97 (Toulouse, August 1997) for helpful comments and discussion. The second author is grateful to the Division of the Humanities and Social Sciences at Caltech for its hospitality during the period in which this paper was written. Ghirardato's e-mail is: `paolo@hss.caltech.edu`. Le Breton's address is CORE, Université Catholique de Louvain, Voie du Roman Pays, 34, 1348 Louvain-La-Neuve, Belgium. E-mail: `lebreton@core.ucl.ac.be`.

DM is known. Besides its theoretical relevance, this "revealed preference" question is interesting for the applications of CEU to studying games in strategic form. In fact, once Choquet rationality is characterized, it is then possible to define and characterize Choquet *rationalizability.*

We completely describe the consequences of the assumption that a DM is a CEU maximizer, first for the special case in which her beliefs are represented by belief functions, and then for the general case in which they can be capacities. Surprisingly, our characterization of Choquet rationality has a strong resemblance to the one obtained for the SEU notion, that we call *Bernoulli rationality.* Precisely: The DM constructs an "extended" decision problem, and then behaves like a SEU maximizer facing that problem. This allows us therefore to rationalize more actions than Bernoulli rationality would.[1] A striking fact is that no additional predictions are obtained by letting the DM's beliefs be represented by capacities which are not belief functions. That is, an action which is rationalized by a capacity can be always rationalized by a belief function. To prove this result we use an altogether new representation of Choquet integrals as additive integrals on an expanded state space, that we believe to be of some separate mathematical interest.

Other papers have looked at revealed preference questions of the sort addressed in this paper, and their application to defining solution concepts for strategic form games. The classical papers for SEU preferences are of course Bernheim [1] and Pearce [26]. There is also a sizable literature focusing of the revealed preference question for SEU preferences (see, e.g., Border [2]). Börgers [3] studies the case in which players are SEU maximizers, but only their ordinal preferences are commonly known (their risk attitude is not). Epstein [11] relaxes the notion of Bernoulli rationality to include a large class of models of preferences, of which SEU is a special case. Using the results of Epstein and Wang [12], he characterizes the consequences of the assumption that a decision model describes all players, and that this is common knowledge among them: The strategies played by the players must be in the set surviving iterated deletion of "dominated" strategies. The notion of dominance, of course, depends on the model. As illustration, he applies his result to some non-expected utility models, for instance the "maxmin expected utility" model of Gilboa and Schmeidler [17] and the "probabilistically sophisticated" model of Machina and Schmeidler [24]. But he does not provide a similar result for the CEU model, since he does not have a characterization of Choquet rationality. Indeed he observes that the latter is an open question for future research.

The structure of the paper is as follows. After introducing the necessary notation and definitions (Section 1), we characterize the actions which are best replies to belief functions, called *Shafer* rational (Section 2). We then (Section 3) allow beliefs to be represented by any capacity, thus characterizing *Choquet* rationality. Section 4 contains some brief remarks on our characterization. Section 5 shows how our result changes if the DM has a preference for randomization. Section 6 analyzes the relation with Börgers's "ordinal rationality" notion, and Section 7 wraps up by presenting Choquet

[1] As we discuss in Section 5, this result depends on whether the DM has a preference for randomization.

rationalizability. The Appendices contain the proofs and technical details. In particular, Appendix A contains the derivation of the additive representation of Choquet integrals.

1 Notation and Preliminaries

We assume that the DM's preferences can be represented by Choquet expected utility maximization on a fully subjective set-up (for axiomatizations, see Gilboa [16] or Wakker [34]). We consider a finite space of *states of the world* Ω, a set of *consequences* $\mathcal{X}$. The choice set $\mathcal{F}$ is a finite set of functions from Ω into $\mathcal{X}$, that we call *acts*.[2]

Given a finite state space Ω, a set-function $\nu : 2^{\Omega} \to \mathbf{R}$ is called a (normalized) *capacity* if it satisfies the following conditions: 1) $\nu(\emptyset) = 0$ and $\nu(\Omega) = 1$; 2) if $A \subseteq B \subseteq \Omega$, then $\nu(A) \leq \nu(B)$. A *belief function* (see Shafer [30]) is a capacity that satisfies the additional condition (called *total monotonicity*): 3) for all n and all collections $A_1, \ldots, A_n \in 2^{\Omega}$,

$$\nu(\cup_{i=1}^{n} A_i) \geq \sum_{\substack{I \subseteq \{1,\ldots,n\} \\ I \neq \emptyset}} (-1)^{\#I+1} \nu(\cap_{i \in I} A_i), \tag{1}$$

where $\#I$ is the cardinality of set I. A capacity which satisfies (1) for pairs of sets ($n = 2$) is called *supermodular* (convex, 2-monotonic). A *probability* is a set-function that satisfies (1) with equality for every collection of sets. Thus a probability is a belief function, but not *vice versa*. We let $\mathcal{V}$ the set of all the capacities and $\mathcal{V}^B$ the (sub) set of all the belief functions on Ω. Finally, given any finite set X, we denote by $\Delta(X)$ the set of the probability measures on $(X, 2^X)$.

Given a function $f : \Omega \to \mathbf{R}$, what is the integral of f with respect to a capacity ν? We use the Choquet integral (see Choquet [5] and Schmeidler [28]), which for the case of finite Ω can be defined as follows: Given f, let $\Omega = \{\omega_1, \ldots, \omega_L\}$, and relabel so that $f(\omega_1) \geq f(\omega_2) \geq \cdots \geq f(\omega_L)$. The *Choquet integral of f with respect to ν* is

$$\int_{\Omega} f(\omega)\, \nu(d\omega) =\equiv \sum_{\ell=1}^{L-1} (f(\omega_\ell) - f(\omega_{\ell+1}))\, \nu(\{\omega_1, \ldots, \omega_\ell\}) + f(\omega_L). \tag{2}$$

It is easy to see that the Choquet integral is monotonic (that is, if $f(\omega) \geq g(\omega)$ for all $\omega \in \Omega$ then for any capacity ν, $\int f\, d\nu \geq \int g\, d\nu$), and when Ω is finite it is also strictly monotonic (if $f(\omega) > g(\omega)$ for all $\omega \in \Omega$, then $\int f\, d\nu > \int g\, d\nu$). The Choquet integral is also seen to be *comonotonic additive*: If $f, g : \Omega \to \mathbf{R}$ are non-negative and comonotonic, then $\int (f+g)\, d\nu = \int f\, d\nu + \int g\, d\nu$. Two functions $f, g : S \to \mathbf{R}$ are said to be *comonotonic* if there are no $\omega, \omega' \in \Omega$ such that $f(\omega) > f(\omega')$ and $g(\omega) < g(\omega')$. A collection $\mathcal{G}$ of fucntions is called *comonotonic* if any two functions in $\mathcal{G}$ are comonotonic.

[2] The restriction to finite Ω is required for the additive decomposition result in Appendix A. Though the extension of that result to infinite Ω appears to be possible, it is beyond the scope of this note. On the other hand, the assumption of finiteness of $\mathcal{F}$ can be easily dispensed with, and it is made only in order to minimize notation and technicalities.

An interesting property of supermodular capacities is the following: Call the *core* of a capacity ν the set $\mathcal{C}(\nu)$ of all the probabilities that dominate ν. Formally,

$$\mathcal{C}(\nu) \equiv \{P \in \Delta(\Omega) : P(A) \geq \nu(A) \text{ for all } A \subseteq \Omega\}.$$

It can be shown (Kelley [20]) that supermodular capacities (and hence belief functions) have a non-empty core, but the converse is not true. Schmeidler [29, Proposition] has shown that for any function $f : \Omega \to \mathbf{R}$, the integral of f with respect to a supermodular ν is equal to the smallest integral with respect to measures in its core $\mathcal{C}(\nu)$. That is,

$$\int_\Omega f(\omega)\,\nu(d\omega) = \min_{P \in \mathcal{C}(\nu)} \int_\Omega f(\omega)\,P(d\omega). \tag{3}$$

The DM has preferences $\succeq$ on $\mathcal{F}$ which are represented as follows: There exists a function $u : \mathcal{X} \to \mathbf{R}$ and a capacity $\nu : 2^\Omega \to [0,1]$ such that for every $f, g \in \mathcal{F}$,

$$f \succeq g \iff \int_\Omega u(f(\omega))\,\nu(d\omega) \geq \int_\Omega u(g(\omega))\,\nu(d\omega),$$

where the integrals are taken in the sense of Choquet. Since the utility function u is kept fixed throughout the analysis, we shall abuse notation and write f in place of $u \circ f$. That is, we henceforth treat acts as real-valued functions.

Following standard terminology for SEU preferences, an act $\hat{f} \in \mathcal{F}$ is said to be *strongly dominated* if there is $\sigma \in \Delta(\mathcal{F} \setminus \{\hat{f}\})$ such that, for every ω,

$$\sum_{f \in \mathcal{F} \setminus \{\hat{f}\}} \sigma(f)\,f(\omega) > \hat{f}(\omega).$$

We also say that $\hat{f}$ is *purely* (strongly) dominated if there is such a σ which is degenerate on some act $f \in \mathcal{F}$. It is widely known that a strategy which is not strongly dominated is the optimal choice for some additive beliefs $P \in \Delta(\Omega)$ (see Proposition 5 in App. B). Hence such a strategy is also called *Bernoulli rational*.

2 Prelude: Shafer Rationality

Suppose that a DM is facing Problem 1 (where the entries are utility payoffs). Assume that $3/2 > \epsilon > 0$. It is easy to see that if the DM chooses an act which maximizes expected utility with respect to *additive* beliefs P on Ω, then she will never choose h. (This can be verified directly, or by noticing that h is strongly dominated by the convex combination of f and g with weight 1/2.) What if, instead, the DM maximizes the Choquet expectation of utility with respect to a *belief function*? The (unsurprising) answer is that then we cannot rule out h. In fact there are belief functions for which h is optimal, for instance the "complete ignorance" belief ν which assigns weight 0 to either

	ω_1	ω_2
f	3	0
g	0	3
h	ϵ	ϵ

PROBLEM 1

ω_1 or ω_2 and 1 to Ω. Since the DM perceives ambiguity (has poor information) about the states ω_1 and ω_2, she prefers the "safe" act h — which assures her a payoff of ϵ whatever happens — to the other "risky" acts.

We are thus led to the revealed preference question: Are there acts that will not be chosen by a DM with CEU preferences? For the case of belief functions, the answer turns out to be straightforward. We first introduce some notation and a definition. We let $\mathcal{N} = 2^\Omega \setminus \{\emptyset\}$, and extend every act f to a function $\bar{f} : \mathcal{N} \to \mathbf{R}$ as follows: for $A \in \mathcal{N}$,

$$\bar{f}(A) = \min_{\omega \in A} f(\omega). \tag{4}$$

The definition extends Bernoulli rationality by allowing the DM's beliefs to be represented by a belief function:[3]

Definition 1 *An act $f^* \in \mathcal{F}$ is called* Shafer rational *if there exists a belief function $\nu \in \mathcal{V}^B$ such that for all $f \in \mathcal{F}$,*

$$\int_\Omega f^*(\omega)\, \nu(d\omega) \geq \int_\Omega f(\omega)\, \nu(d\omega).$$

Act f^* is called *Shafer dominated* if it is not Shafer rational. Using game-theoretic terminology: Act f^* is Shafer rational if it is a *best reply* to some belief (function). We can now answer the question posed above by *characterizing* Shafer rationality.

Theorem 1 *Act f^* is Shafer rational* if and only if *there does not exist $\sigma \in \Delta(\mathcal{F} \setminus \{f^*\})$ such that for all $A \in \mathcal{N}$,*

$$\sum_{f \in \mathcal{F} \setminus \{f^*\}} \sigma(f)\, \bar{f}(A) > \bar{f}^*(A). \tag{5}$$

This result (which is Cor. 2 in App. B) is an immediate consequence of the characterization of Bernoulli rationality, and a widely known representation of Choquet integrals with respect to belief functions as (additive) integrals with respect to probabilities on a larger state space (Cor. 1 in App. A.2).

[3] Recall that the DM's utility function is assumed to be known and fixed in the analysis.

	ω_1	ω_2	$\{\omega_1, \omega_2\}$
f	3	0	0
g	0	3	0
h	ϵ	ϵ	ϵ

Table 1: The extension of Problem 1

3 Choquet Rationality

The characterization of Theorem 1 is surprisingly simple and appealing, as it bears a lot of resemblance with that of Bernoulli rationality: A DM maximizing a Choquet integral with respect to a belief function behaves like a SEU maximizer who faces a problem with an "extended" state space $2^\Omega \setminus \{\emptyset\}$, where acts are extended in the "pessimistic" manner described by Eq. (4).[4] For instance, in Problem 1, it is as if a SEU DM were looking at the problem given by Table 1. Here h is not strongly dominated, so Theorem 1 confirms what we argued above: There is a belief function which rationalizes the choice of h.

On the other hand, it has certainly not escaped the reader's attention that we have considered only a very special class of capacities, namely belief functions. It is reasonable to conjecture that if we allowed the DM's beliefs to be an arbitrary capacity, we would be able to rationalize *more* acts, thus obtaining a weaker prediction. This is also suggested by the fact that belief functions embody (as shown by Ghirardato and Marinacci [15]) a form of "ambiguity aversion": One would expect the picture to change once we allow capacities which entail "ambiguity loving" behavior.

The main finding of this note is that this conjecture is surprisingly wrong. To present this result, we start by extending of Definition 1, so as to allow the DM's beliefs to be represented by an arbitrary capacity.

Definition 2 *An act $f^* \in \mathcal{F}$ is called* Choquet rational *if there exists a capacity $\nu \in \mathcal{V}$ such that for all $f \in \mathcal{F}$,*

$$\int_\Omega f^*(\omega)\,\nu(d\omega) \geq \int_\Omega f(\omega)\,\nu(d\omega).$$

Clearly, if an act is Shafer rational, it is also Choquet rational. What is less obvious is that the converse is also true.

Theorem 2 *Act $f^* \in \mathcal{F}$ is Choquet rational* if and only if *it is Shafer rational.*

[4] The technique of looking at this extended decision problem for describing best replies to belief functions is not new. Hendon *et al.* used it in [19], where they discuss a notion of equilibrium for games in which players' beliefs can be belief functions.

In other words: An act is a best reply to a capacity if and only if it is a best reply to a belief function. The proof of this result uses a different additive representation of the Choquet integral which is, to the best of our knowledge, new. It is presented in Appendix A (Prop. 3). The proof of Theorem 2 is found in Appendix C, but the intuition is very simple: For any capacity ν rationalizing act f^*, one can use its additive representation to explicitly construct a belief function ν' which also rationalizes f^*.

Theorem 2 shows that, without more information on the DM's beliefs (see Remark 4.1 below), Shafer rationality is the *only* empirical implication of the assumption that the DM has CEU preferences with utility u. That is, the extension from belief functions to capacities has no empirical relevance.

It is important to remark, though, that sharper predictions could be made if it were possible to observe the DM's choices on different subsets of $\mathcal{F}$. For instance, consider Problem 1 with $\epsilon = 2$, and suppose that the DM has beliefs given by the capacity $\nu(\{\omega_1\}) = 3/4$, $\nu(\{\omega_2\}) = 3/4$. It is easy to verify that ν is not supermodular, hence not a belief function. If faced with the restricted choice set $\{g, h\}$, the DM will choose g, and if faced with $\{f, h\}$, she will choose f. But these choices cannot be made by a CEU DM whose capacity is a belief function.[5] Hence, being able to observe the DM's revealed preferences over multiple choice sets enables us to impose tighter constraints on her beliefs.

4 Remarks

4. 1 It follows from Theorem 2 that we obtain the same notion of rationality as long as we require the DM's beliefs to belong to any class of capacities containing the set $\mathcal{V}^B$. (This will be the case, for instance, if we assume that the DM is ambiguity averse in the sense of [15].)

However, this fails to be true if we consider sets of beliefs which are proper subsets of $\mathcal{V}^B$. For instance, having seen how we justify the DM's choice of h in Problem 1, one could wonder whether the additional predictive power of Choquet rationality is only due to "completely ignorant" beliefs, like the one which is 0 everywhere but on every event implied by some $A \subseteq \Omega$. These are belief functions which are usually called "unanimity games" (see App. A.2), and they entail a (constrained) "maxmin" rule: An act is not ruled out if it is a maxmin choice over A. The answer is no: There are problems in which

[5] To see why, recall first that if the DM's beliefs are represented by a belief function ν', the core $\mathcal{C}(\nu')$ is non-empty and Eq. (3) applies. The choice of f over h implies that there is a probability $P \in \mathcal{C}(\nu')$ such that $P(\{\omega_1\}) > 2/3$. The choice of g over h that there is $P' \in \mathcal{C}(\nu')$ such that $P'(\{\omega_2\}) > 2/3$, or equivalently $P'(\{\omega_1\}) < 1/3$. But by (3) we then have

$$\int_\Omega f \, d\nu' \leq \int_\Omega f \, dP' < 1,$$

so that f cannot be preferred over h (whose expectation is 2). This provides the required contradiction.

an act is strongly dominated and Choquet rational, and it cannot be rationalized by a belief function with this degenerate form.[6] ◇

4. 2 Problem 1 proves that Choquet dominance is strictly weaker (rules out fewer acts) than strong dominance. Moreover, it is strictly stronger than pure dominance. In fact, it follows from the monotonicity of Choquet integrals that a strategy which is purely dominated is Choquet dominated, while the converse is false. For example, consider the following decision problem:

	ω_1	ω_2
f	8	0
g	4	4
h	5	1

Problem 2

Clearly no act is purely dominated. By constructing the extended problem one can check that h is strongly dominated by the convex combination of f and g with weight $1/2$, hence it is Choquet dominated. ◇

4. 3 A model different from CEU, which also allows ambiguity to matter in decision making is the *maxmin expected utility* (MEU) model of Gilboa and Schmeidler [17]. In this model beliefs are represented by a non-empty closed and convex set of probabilities $\mathcal{C}$, and an act $f \in \mathcal{F}$ is evaluated via the functional

$$\min_{P \in \mathcal{C}} \int_\Omega f(\omega)\, P(d\omega).$$

The two models are not nested, i.e., there are CEU DMs which cannot be described by MEU and *vice versa*. However, by Eq. (3) the DMs whose preferences are described by CEU with supermodular beliefs satisfy both models.

By Theorem 2, every Choquet rational act can be "rationalized" by a belief function, which is supermodular. Hence, a Choquet rational act is also "MEU rational" (for a characterization, see [11]). To see that the converse does not hold, consider Problem 3 (which is a modification of an example of Klibanoff in [22]). Let $\mathcal{C}$ be the closed and

[6] For instance, consider the following variation on Problem 1:

	ω_1	ω_2
f	3	1.1
g	0	3
h	1	2

h is Choquet rational, but it cannot be rationalized by a unanimity game (enough positive weight on ω_2 is needed to make f sufficiently bad).

	ω_1	ω_2	ω_3
f	1.5	2	3.5
g	0	2.1	4
h	0.7	2	3.7

Problem 3

convex class of probabilities on Ω defined as the convex hull of $P_1 = (0, 1, 0)$ and $P_2 = (1/2, 0, 1/2)$. It is straightforward to verify that $\mathcal{C}$ rationalizes the choice of h. On the other hand, Choquet dominance rules h out.[7] In conclusion: CEU maximizing behavior makes *sharper* predictions than the maximization of the multiple priors functional. $\diamond$

5 Choquet Rationality with Randomizations

In this section we discuss the consequences of the assumption of Choquet rationality in the case in which: 1) there is an unambiguous (additive) randomizing device that the DM can use to actively randomize her choice over acts; 2) the DM employs the device, so that she considers $\Delta(\mathcal{F})$ to be her choice set, rather than $\mathcal{F}$. This is relevant especially in view of the application of our results to the solution of strategic form games in Section 7.

The presence of a randomizing device does not change the characterization of Bernoulli rationality. In fact, if a "mixed" act (a nondegenerate probability in $\Delta(\mathcal{F})$) is optimal with respect to some additive beliefs, then every "pure" act (a degenerate probability in $\Delta(\mathcal{F})$) in the support of the randomization is also optimal. Hence, the set of Bernoulli rational strategies when randomization is allowed is the set of all the randomizations over the Bernoulli rational pure acts.[8] However, it is widely believed that ambiguity averse CEU preferences necessarily embody preference for randomization. That is, a mixed act might be preferred by a CEU DM to every pure act in its support, so that she would strictly prefer to make use of a randomizing device, if available. Therefore, the analysis of Choquet rationality might change sensibly.

The fact that ambiguity aversion gives rise to preference for randomization has been debated in the literature on decision making. Indeed, preference for randomization appears in the Anscombe-Aumann framework used by Schmeidler [29], with his notion of ambiguity aversion (which is equivalent to supermodularity of beliefs). However this result has some problems. First of all, recent work by Epstein [10] and Ghirardato and

[7] Indeed, f, g and h are comonotonic, and the following claim is easily verified: *For a comonotonic set of acts $\mathcal{F}$, strong dominance and Choquet dominance are equivalent.* Also, notice that the example is not knife-edge: We could even make h strictly better for the multiple priors DM by adding an $\epsilon \in (0, 0.05)$ to all its entries.

[8] The point of view that players in a game might actively randomize is increasingly criticized by game theorists (see, e.g., the discussion in Osborne and Rubinstein [25, pp. 37–45]).

Marinacci [15] has cast some doubts on whether ambiguity aversion is indeed characterized by supermodularity. More crucially, Schmeidler's result builds on the two-stage structure of the Anscombe-Aumann model. That is, it is assumed that the DM will evaluate act $\sigma \in \Delta(\mathcal{F})$ according to the functional

$$\int_{\Omega} \left(\sum_{f \in \mathcal{F}} \sigma(f)\, f(\omega) \right) \nu(d\omega). \tag{6}$$

Differently from the additive case, though, the order of integration matters for Choquet integrals. Hence (6) is in general *not* the same as the reverse functional

$$\sum_{f \in \mathcal{F}} \sigma(f) \left(\int_{\Omega} f(\omega)\, \nu(d\omega) \right). \tag{7}$$

In other words, the logical order in which the state of nature and the result of the randomization are decided matters. Eq. (6) evaluates mixed acts *as if* the state is decided first, and then the randomization is performed. Eq. (7) does the opposite. Preference for randomization occurs for supermodular capacities when (6) is how the DM judges mixed acts. It does not, so that the analysis of the previous sections is valid, if (7) is used instead: If f and f' are pure acts that are indifferent, then clearly any randomization over them will have the same utility. It is not obvious that (6) has any conceptual precedence over (7).

An interesting way to avoid the arbitrariness in this choice, suggested by Eichberger and Kelsey [8] (EK), is to study the existence of preference for randomization in a one-stage approach: Build the randomizing device into the framework (by enlarging the state space with the outcomes of the randomization), and assume that randomizing device and states of nature are stochastically independent. Using a weak notion of (stochastic) independence, EK show that a DM with supermodular beliefs is *indifferent* to randomization.[9] And it is possible to use a result of Ghirardato [14] to show that if a stronger notion of independence is used, a similar result holds for general CEU preferences. Thus it seems that (if independence is properly modelled) no preference for randomization can appear in the one-stage case.

Setting these doubts aside, we now show the consequences of Choquet rationality if the DM's strategy set is $\Delta(\mathcal{F})$, and she evaluates mixed acts according to (6).[10] Suppose that act $\sigma \in \Delta(\mathcal{F})$ is (strongly) dominated. Then there is $\sigma^* \in \Delta(\mathcal{F})$ such that for every $\omega \in \Omega$

$$\sum_{f \in \mathcal{F}} \sigma^*(f)\, f(\omega) > \sum_{f \in \mathcal{F}} \sigma(f)\, f(\omega). \tag{8}$$

[9] However there is a problem with using this one-stage construction and assuming supermodularity. Klibanoff [23] shows that an intuitive strengthening of EK's independence condition then forces the DM's beliefs to be be *additive*.

[10] Obvious symmetry consideration suggest that it would be also appropriate to extend our analysis to the case in which Ω is infinite (also the other player can play a mixed strategy!). This requires extending the results in App. A to the infinite case. Such extension seems possible, but it goes beyond the scope of this paper, without changing the substance of our results.

For any mixed act $\sigma \in \Delta(\mathcal{F})$, define $\bar{\sigma}$ on $\mathcal{N}$ as follows: For every $A \in \mathcal{N}$,

$$\bar{\sigma}(A) \equiv \min_{\omega \in A} \sigma(\omega) \equiv \min_{\omega \in A} \sum_{f \in \mathcal{F}} \sigma(f)\, f(\omega).$$

Using (8), it is immediate to see that in particular we have that for every $A \in \mathcal{N}$,

$$\bar{\sigma}^*(A) > \bar{\sigma}(A). \tag{9}$$

That is, the extension of σ^* (strongly) dominates the extension of σ on $\mathcal{N}$. It is now simple to see that σ^* Shafer dominates σ, so that σ cannot be Shafer rational.[11] A straightforward modification of Theorem 2 then implies that σ cannot be Choquet rational. We have thus proved:

Proposition 1 *Suppose that the DM's choice set is given by $\Delta(\mathcal{F})$, and randomizations are evaluated by Eq. (6). Then a (mixed) act $\sigma \in \Delta(\mathcal{F})$ is Choquet rational* if and only if *it is not (strongly) dominated.*

Thus, under the conditions of the Proposition, Choquet rationality and Bernoulli rationality coincide.[12]

However, it is probably worth closing by reiterating that this is not going to be true if we model (the DM's evaluation of) randomized acts according to the opposite two-stage structure, or a one-stage structure (under suitable independence conditions). In that case the set of Choquet rational acts is the set of the randomizations over the *pure* Choquet rational acts, as described by Theorem 1.

6 Ordinal Rationality

6.1 Risk and Ambiguity: Ordinal Rationality and Choquet Rationality

As mentioned in the Introduction, Börgers [3] studies the case in which, though the DM's ranking of the possible consequences of her decisions is known, we do not know her risk attitude (that is, her cardinal utility index). He considers a SEU DM who has a given preference ordering over consequences, and asks which acts she might play. That is, which acts are best replies to some (additive) belief, and some cardinal utility index representing her (given) preferences over the possible outcomes.[13] We call such acts *ordinally rational.*

[11] This follows from Eq. (13) in App. A.2: Given that (9) holds, for every probability on $\mathcal{N}$, the expectation of the LHS is strictly greater than that of the RHS. But each probability maps to a belief function, and each such expectation equals the Choquet integral with respect to some belief function.

[12] A similar result was proved by Klibanoff [21] for the multiple priors model.

[13] As usual, we say that a function $u : \mathcal{F} \times \Omega \to \mathbf{R}$ *represents* $\succeq$ if for every $(f, \omega), (f', \omega') \in \mathcal{F} \times \Omega$,

$$f(\omega) \succeq f'(\omega') \Longleftrightarrow u(f(\omega)) \equiv u(f, \omega) \geq u(f', \omega') \equiv u(f'(\omega')).$$

In this first subsection we discuss the relation between ordinal rationality and Choquet rationality. Precisely: Suppose that we observe a DM facing a given decision problem (with cardinal payoffs given by a function u). Will the predictions obtained by varying her ambiguity attitude be different from those obtained by varying her risk attitude? As we show by example, the answer is yes. This provides an interesting qualitative difference between ambiguity aversion and risk aversion.

Given an act $f \in \mathcal{F}$, we say that it is *ordinally dominated* if for every subset $A \subseteq \Omega$, there is an act $f_A \in \mathcal{F} \setminus \{f\}$ such that f_A weakly dominates f on A. That is, for every $\omega \in A$,

$$f_A(\omega) \succeq f(\omega),$$

with strict inequality for some $\omega \in A$. Börgers shows that ordinal rationality can be characterized by ordinal dominance as follows:

Proposition 2 (Börgers [3]) *An act f is ordinally rational* if and only if *it is not ordinally dominated.*

For example, consider Problem 1 with $\epsilon = 1$: Act h is ordinally rational, since neither f nor g weakly dominate h on the subset Ω. (This is also seen by considering the monotonic transformation of payoffs given by φ, where $\varphi(3) = 3$, $\varphi(1) = 2$, $\varphi(0) = 0$.)

To see that Choquet rationality and ordinal rationality give different predictions, consider first Problem 1 with $\epsilon = 0$. Act h is ordinally dominated (by either of the other acts on Ω, by f on $\{\omega_1\}$, by g on $\{\omega_2\}$), but it is not Choquet dominated (it can be rationalized by the "completely ignorant" capacity which is 0 except on Ω). The intuition is the following: Even if Row's risk attitude is not correctly reflected by the numbers in the game table, (if she is ambiguity neutral) she will not choose h, since whatever her beliefs she will be better off by choosing either of the other acts (in this problem ordinal dominance does not have additional predictive power over standard dominance). On the other hand, she might choose h if she is extremely ambiguity averse, thinking that whatever she does, the worst possible state will obtain. Then she is *indifferent* over all her acts.

The opposite can also happen. Consider Problem 2: As explained in Remark 2, there h is not Choquet rational since the convex combination of f and g with weight 1/2 dominates h in the extended problem. However, h is ordinally rational, since no other act weakly dominates it on Ω (alternatively, transform the payoff which is now 4 to a 2, and check that no convex combination of f and g dominates h). The point here is that f, g and h are similar in their riskiness (they are comonotonic), so that CEU does not really buy anything more than SEU (cf. footnote 7): If the DM is sufficiently confident that the state is ω_1, she chooses f, and if she doesn't, she is better off by playing the safe option g. On the other hand, interpret the numbers in the decision table as monetary payoffs and assume that the DM is very risk averse around 0, but relatively risk neutral between 1 and 8. Then if she assigns a probability to ω_1 which is large but not too large, she might prefer h to both the other acts.

6.2 Ordinal Choquet Rationality

In this subsection we extend Börgers's analysis [3] to the CEU case. That is, we ask which predictions can be made if we only know the DM's ordinal ranking $\succeq$ of outcomes $(f, \omega) \in \mathcal{F} \times \Omega$, but do not know her u. The following definition adapts Börgers's definition to the case in which the DM is a CEU maximizer whose beliefs are represented by a belief function.

Definition 3 *An act $f^* \in \mathcal{F}$ is called* ordinally Shafer rational *if there exists a utility function u representing $\succeq$ and a belief function $\nu \in \mathcal{V}^B$ such that for all $f \in \mathcal{F}$,*

$$\int_\Omega u(f^*(\omega))\, \nu(d\omega) \geq \int_\Omega u(f(\omega))\, \nu(d\omega).$$

Extend $\succeq$ to a complete preorder $\succcurlyeq$ on $\mathcal{F} \times \mathcal{N}$ as follows: For any $A, A' \in \mathcal{N}$ and $f, f' \in \mathcal{F}$ let $(f, A) \succcurlyeq (f', A')$ if

$$\min_{\omega \in A} (f, \omega) \succeq \min_{\omega' \in A'} (f', \omega'),$$

where $\min_{\omega \in A} (f, \omega)$ is a $\succeq$-minimal element of $\{f\} \times A$. As usual, we denote by $\succ$ the asymmetric component of $\succcurlyeq$.

Theorem 3 *Act $f^* \in \mathcal{F}$ is ordinally Shafer rational* if and only if *for every non-empty subset $\mathcal{A} \subseteq \mathcal{N}$ there is no $f \in \mathcal{F}$ such that for all $A \in \mathcal{A}$,*

$$(f, A) \succcurlyeq (f^*, A), \tag{10}$$

with $\succ$ holding for at least one $A \in \mathcal{A}$.

As we did previously, we will call *ordinally Shafer dominated* an act f^* such that for every $\mathcal{A} \subseteq \mathcal{N}$ there is $f \in \mathcal{F}$ for which (10) holds. The proof of the result is a simple adaptation of the proof of Theorem 1, and it is given in Appendix D. The next step (omitted for brevity) is to observe that a result like Theorem 2 holds also in this case, showing that we add no generality by defining ordinal *Choquet* rationality. In other words, ordinal Shafer dominance completely characterizes the behavior of a CEU maximizer whose preference ranking is given by $\succeq$. Hence we can equivalently talk about ordinal Choquet dominance (*pace* Shafer).

As we had above, it is immediate to notice that if an act is purely dominated, then it is ordinally Choquet dominated. To see that the converse is not true, consider Problem 4. Assume that the DM ranks the "physical" outcomes as follows: $a \succ b \succ c$. Clearly h is not strictly dominated by f or g. The extended decision problem is depicted in Table 2. With a little work one can then use the table to check that h is ordinally Choquet dominated.

	ω_1	$\omega 2$
f	a	c
g	b	b
h	b	c

PROBLEM 4

	ω_1	ω_2	$\{\omega_1, \omega_2\}$
f	a	c	c
g	b	b	b
h	b	c	c

Table 2: The extension of Problem 4

Finally, Problem 1 with $\epsilon = 0$ shows that ordinal rationality is more selective than ordinal Choquet rationality: As pointed out before, ordinal rationality rules out h, but (unsurprisingly) ordinal Choquet rationality does not (consider the family $\mathcal{A} = \{\{\omega_1, \omega_2\}\}$). However, the considerations made above to show that Choquet rationality is almost more restrictive than ordinal rationality can be repeated here, to show that ordinal Choquet dominance does not buy much more than ordinal rationality. Indeed, h will never be a strict best reply to a capacity ν. That is, for any utility function u representing $\succeq$ and any capacity ν, both f or g will always be at least as good as h. This is true in general, as it is an immediate consequence of the monotonicity of Choquet integrals that: If Börgers dominance results in a smaller set than ordinal Choquet dominance, for every ordinally Choquet undominated act there is *some* Börgers undominated act indifferent to it (with respect to the rationalizing u and ν). But this is no surprise, since we have seen in the previous subsection that Choquet rationality allows predictions which are ruled out by ordinal rationality only under these circumstances.

7 An Application: Choquet Rationalizatibility

A natural application of our characterization result is to the definition of a solution concept for strategic-form n-person games, that we call ("correlated") *Choquet rationalizability*. In the same spirit as Bernheim [1] and Pearce [26]'s solution concept, one can ask which subset of strategy n-tuples is the largest Cartesian set with the following property: Every strategy for player i included in the set is a best reply to a capacity which reflects player i's belief that the other players will also play a strategy in the set. For a strong notion of "belief", the exercise is straightforward, since we can make use of a general characterization result of Epstein [11]. We thus content ourselves with giving the necessary definitions and stating the results. Given a DM with beliefs $\nu \in \mathcal{V}$ on the state space Ω, we say that the DM *believes* an event $E \subseteq \Omega$ if $\nu(E^c \cup F) = \nu(F)$ for every $F \subseteq \Omega$.[14] Notice that when ν is not additive this is stronger than requiring $\nu(E^c) = 0$.

Consider now the decision problem faced by each player $i \in I$ in the strategic form game $\Gamma = (I, (S_i)_{i \in I}, (u_i)_{i \in I})$,[15] where $\Omega = S_{-i} \equiv \times_{j \neq i} S_j$. We say that a set $R = R_1 \times \ldots \times R_I$ of strategy profiles is the *Choquet rationalizable* set if it is the largest Cartesian subset of $\mathbf{S} = \times_{i \in I} S_i$ with the property: For every $i \in I$ and every $s_i \in R_i$, s_i is a best reply to some capacity ν_i on S_{-i} which believes $R_{-i} \equiv \times_{j \neq i} R_j$. In other words, every strategy $s_i \in R_i$ is Choquet rational for the state space R_{-i}. It follows from [11, Theorem 3.2] that the Choquet rationalizable set is nonempty and it coincides with the unique result of the procedure of iterated deletion of Choquet dominated strategies.[16] For instance, applying this result to the following game (which is based on Problem 3) we find that the Choquet rationalizable set is {M} × {R}. Clearly, if the conditions

	L	C	R
T	1.5 , -1	2 , 1	3.5 , 0
M	0 , -1	2.1 , 0	4 , 1
B	0.7 , 2	2 , 0	3.7 , -1

A Game

of Proposition 1 hold instead, Choquet rationalizability and (Bernoulli) rationalizability coincide.

Finally we have the epistemic characterization of Choquet rationalizability. We can use the type space discussed in [11] to define the event that each player $i \in I$ is Choquet rational (i.e., plays a Choquet rational strategy), and that this is common belief among all players. The desired result then follows from [11, Theorem 6.3]: *A strategy profile s is Choquet rationalizable if and only if all players are Choquet rational and this is common belief.*

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[14] In decision theoretic jargon: the DM believes E if E^c is *Savage-null* (see [11]; for a proof of the equivalence see Schmeidler [29, Remark 4.3]). This notion of belief is quite strong. For instance, a popular notion of equilibrium for games with CEU players (Dow and Werlang's [7] *Nash equilibrium with Knightian uncertainty*) predicts outcomes which are ruled out by Choquet rationalizability.

[15] As usual, each strategy set is finite, I is finite, and $u_i : S \to \mathbf{R}$

[16] As in Epstein's more general case, the order of deletion of Choquet dominated strategies does not matter.

Appendix A Additive Representations of Choquet Integrals

Here we discuss how to represent Choquet integrals as standard integrals on a state space which is strictly larger than Ω. There is already a widely known result of this form that represents (for a finite Ω) a Choquet integral with respect to a capacity ν as an integral with respect to a *signed* measure on the space 2^Ω (see Corollary 1 below). We provide here a different decomposition which, even though it uses an even larger state space and is not unique, has the great advantage from our perspective of always expressing a Choquet integral as an integral with respect to a probability measure.

To avoid technical difficulties that would bring us beyond the scope of this paper, we henceforth restrict our attention to the case in which Ω is finite. We believe, however, that a similar decomposition can be obtained also for infinite state spaces.

A.1 Capacities

Let $\mathcal{V}$ be the set of all the normalized capacities on Ω. Assuming that $\#\Omega = L$ we have that $\mathcal{V} \subseteq \mathbf{R}_+^K$, where $K = 2^L$. As $\mathcal{V}$ is obviously convex, closed and bounded, it is the convex hull of its extreme points (for a proof see, e.g., Rockafellar [27, Corollary 18.5.1]). The latter are all the capacities ν such that $\nu(A) = 0$ or $\nu(A) = 1$ for all $A \subseteq \Omega$, the $\{0,1\}$-valued capacities (for a proof see Choquet [5, Theorem 40.1]). We let $\mathcal{E} \equiv \{e_1, \ldots, e_N\}$ be the collection of all such $\{0,1\}$-valued capacities (clearly $2^{K-2} \geq N \geq K-1$). Given $e \in \mathcal{E}$ consider the family $\mathcal{A}_e$ of subsets which are assigned weight 1 by e. Let $\mathcal{B}_e$ be the set of all the elements of $\mathcal{A}_e$ which are minimal with respect to set inclusion. That is,

$$\mathcal{B}_e \equiv \{A \subseteq \Omega : e(A) = 1, \, \forall B \subset A, \, e(B) = 0\}.$$

Clearly the mapping $e \mapsto \mathcal{B}_e$ gives a bijection between the set $\mathcal{E}$ and the set $\mathbf{B}$ of all the classes $\mathcal{B}$ of subsets of Ω satisfying the property that if $A, B \in \mathcal{B}$, then *both* $A \setminus B \neq \emptyset$ and $B \setminus A \neq \emptyset$. We then have an interesting representation of Choquet integrals:

Proposition 3 *For every capacity ν on $(\Omega, 2^\Omega)$ there is a probability measure $\alpha(\nu)$ on $(\mathcal{E}, 2^\mathcal{E})$ such that for every $f : \Omega \to \mathbf{R}$, the Choquet integral of f w.r.t. ν admits the following additive decomposition:*

$$\int_\Omega f(\omega)\, \nu(d\omega) = \sum_{n=1}^N \alpha_n(\nu) \max_{A \in \mathcal{B}_n} [\min_{\omega \in A} f(\omega)], \tag{11}$$

where we let $\mathcal{B}_n = \mathcal{B}_{e_n}$ and $\alpha_n = \alpha(\nu)(e_n)$.

Proof: Let $f : \Omega \to \mathbf{R}$. We first show that the Choquet integral of f w.r.t. a capacity $e \in \mathcal{E}$ can be written as follows

$$\int_\Omega f(\omega)\, e(d\omega) = \max_{A \in \mathcal{B}_e} [\min_{\omega \in A} f(\omega)], \tag{12}$$

To see why this is true, start by relabelling the elements of Ω so that $f(\omega_1) \geq \cdots \geq f(\omega_L)$. Using the definition of the Choquet integral in (2) we have that

$$\int_\Omega f(\omega)\, e(d\omega) = f(\omega_\ell),$$

where ℓ is the smallest integer such that $e(\{\omega_1, \ldots, \omega_\ell\}) = 1$. Since $f(\omega_\ell) = \min_{1\leq i\leq \ell} f(\omega_i)$, we only need to show that $\min_{1\leq i\leq \ell} f(\omega_i) = \max_{A\in\mathcal{B}_e}[\min_{\omega\in A} f(\omega)]$. If that were not the case then there would be an $A \in \mathcal{B}_e$ such that $A \subseteq \{\omega_1, \ldots, \omega_{\ell-1}\}$, which is impossible by the definition of the index ℓ. This concludes the proof of (12).

Using (12), it is immediate to obtain the additive decomposition in (11). Since $\mathcal{V}$ is the convex hull of $\mathcal{E}$, for any $\nu \in \mathcal{V}$ there is (at least one) vector $\alpha(\nu) = [\alpha_1(\nu), \ldots, \alpha_N(\nu)]$, satisfying $\alpha_n(\nu) \geq 0$ for every n and $\sum_n \alpha_n(\nu) = 1$, such that we can write $\nu = \sum_n \alpha_n(\nu)\, e_n$. Clearly the vector $\alpha(\nu)$ can be extended to a probability on $2^{\mathcal{E}}$, which we will denote identically.

We now recall the simple fact that the Choquet integral is linear in the integrating capacity to obtain

$$\begin{aligned}
\int_\Omega f(\omega)\, \nu(d\omega) &= \int_\Omega f(\omega)\, [\sum_{n=1}^N \alpha_n(\nu) e_n](d\omega) \\
&= \sum_{n=1}^N \alpha_n(\nu) \int_\Omega f(\omega)\, e_n(d\omega), \\
&= \sum_{n=1}^N \alpha_n(\nu) \max_{A\in\mathcal{B}_n} [\min_{\omega\in A} f(\omega)],
\end{aligned}$$

which is what we wanted to show. ∎

Notice that the additive decomposition is not in general unique. That is, there can be multiple measures $\alpha(\nu)$ for which (11) holds for every f. This is immediate from the proof of Proposition 3: Each $\alpha(\nu)$ corresponds to one of the (in general multiple) ways to express ν as a convex combination of the extreme points $e \in \mathcal{E}$.

A.2 Belief Functions

Let $\mathcal{V}^B$ be the subset of $\mathcal{V}$ containing all the belief functions. Clearly every $\nu \in \mathcal{V}^B$ is a convex combination of points of $\mathcal{E}$, as described above. We will now show that every such ν can be expressed as a unique convex combination of extreme points belonging to a subset $\mathcal{E}^B \subseteq \mathcal{E}$, the set of all the *unanimity games*. A capacity e is a unanimity game if there is $A \subseteq \Omega$ such that $e(B) = 1$ if $A \subseteq B$ and $e(B) = 0$ otherwise. We use the notation e_A for the unanimity game on set $A \subseteq \Omega$. Clearly every unanimity game is $\{0, 1\}$-valued, hence an element of $\mathcal{E}$, and it is easy to check that it is a belief function.

Let $\mathcal{N} = 2^\Omega \setminus \emptyset$ and $\mathcal{E}^B \equiv \{e_A : A \in \mathcal{N}\}$. Indeed, one can show that the set $\mathcal{E}^B$ can be used to generate by *linear* combination any $\nu \in \mathcal{V}$, and by *convex* combination any $\nu \in \mathcal{V}^B$. This is the point of the following result due to Shapley [32].

Proposition 4 *The family $\mathcal{E}^B$ forms a linear basis for $\mathcal{V}$. That is, for any $\nu \in \mathcal{V}$ there is a vector $\varphi(\nu) \in \mathbf{R}^{(K-1)}$ such that*

$$\nu = \sum_{A \in \mathcal{N}} \varphi(\nu)(A)\, e_A.$$

In particular $\varphi(\nu)$ is non-negative if and only if $\nu \in \mathcal{V}^B$.

The last statement implies our claim that the set $\mathcal{V}^B$ is equal to the convex hull $\mathcal{E}^B$. In terms of the previous subsection, we have that a decomposition of $\nu \in \mathcal{V}^B$ over $\mathcal{V}$ is the one with weights given by $\alpha(\nu)$, where $\alpha(\nu)(e) = \varphi(\nu)(A)$ for every $e = e_A$, $A \in \mathcal{N}$, and $\alpha(\nu)(e) = 0$ for $e \in \mathcal{E} \setminus \mathcal{E}^B$. The function $\varphi(\nu)$ is called the *Möbius transform* of ν. We thus immediately have the following:

Corollary 1 *Given $\nu \in \mathcal{V}^B$, there is a unique probability measure $\varphi(\nu)$ on $(\mathcal{E}^B, 2^{\mathcal{E}^B})$ such that for every $f : \Omega \to \mathbf{R}$, the Choquet integral of f w.r.t. ν admits the following additive decomposition:*

$$\int_\Omega f(\omega)\, \nu(d\omega) = \sum_{A \in \mathcal{N}} \varphi(\nu)(A)\, [\min_{\omega \in A} f(\omega)]. \tag{13}$$

Proof: Once again, start by observing that the Choquet integral of a function $f : \Omega \to \mathbf{R}$ with respect to a unanimity game e_A, $A \subseteq \Omega$, is

$$\int_\Omega f(\omega)\, e_A(d\omega) = \min_{\omega \in A} f(\omega). \tag{14}$$

This follows immediately from (12) and the observation that the class $\mathcal{B}_{e_A}$ contains only the set A. Now, given any $\nu \in \mathcal{V}^B$, the observation following Proposition 4 proves that we can rewrite (11) as follows:

$$\begin{aligned} \int_\Omega f(\omega)\, \nu(d\omega) &= \int_\Omega f(\omega)\, [\sum_{n=1}^N \alpha_n(\nu) e_n](d\omega) \\ &= \sum_{A \in \mathcal{N}} \varphi(\nu)(A) \int_\Omega f(\omega)\, e_A(d\omega), \\ &= \sum_{A \in \mathcal{N}} \varphi(\nu)(A)\, [\min_{\omega \in A} f(\omega)], \end{aligned}$$

which is what we had to prove. ∎

As we observed at the beginning of this appendix, the decomposition in (13) is widely known. It follows immediately from Proposition 4 that it can be extended to any capacity ν, with the important *caveat* that then $\varphi(\nu)$ is a probability if and only if ν is a belief function (in general $\varphi(\nu)$ can take negative values).[17] We chose to state and prove the Corollary in this way to show the connection between the usual decomposition result and that contained in Proposition 3. We also remark that in this case uniqueness of $\varphi(\nu)$ obtains because the "extended state space" $\mathcal{N}$ has the same cardinality as the set over which ν can take positive values (which is $\mathcal{N}$ itself).

Appendix B Some Domination Results

Let S and T be finite sets with $\#S = M$ and $\#T = N$. Let $U : S \times T \to \mathbf{R}$. The following well-known result is due to Van Damme [33] and Pearce [26]. It can easily be proved using a result on linear inequalities (see, e.g., Gale [13, Theorem 2.10]).

Proposition 5 *Let $s^* \in S$ and $S' = S \setminus \{s^*\}$. Exactly one of the following alternatives holds:*

1. Either *there does exist an $\alpha \in \Delta(T)$ such that for all $s \in S$,*

$$\sum_{t \in T} \alpha(t)\, U(s^*, t) \geq \sum_{t \in T} \alpha(t)\, U(s, t), \tag{15}$$

   where $\alpha(t)$ is the probability that α assigns to $t \in T$;

2. or *there is a $\sigma \in \Delta(S')$ such that for all $t \in T$*

$$\sum_{s \in S'} \sigma(s)\, U(s, t) > U(s^*, t). \tag{16}$$

Using the decision-theoretic terminology: An act is a best reply to some (additive) belief iff it is not strongly dominated.

It is immediate to use the additive decomposition results in the previous appendix to obtain as corollaries of Proposition 5 domination results for the case in which the players are CEU maximizers. For instance, we have the following characterization of best replies to belief functions:

Corollary 2 *Let $\mathcal{F}$ and Ω be finite, and $u : \Omega \to \mathbf{R}$. Let $f^* \in \mathcal{F}$ and $\mathcal{F}' = \mathcal{F} \setminus \{f^*\}$. Exactly one of the following alternatives holds:*

[17] For finite state space, the result was first proved for belief functions by Shafer [31], and extended to supermodular capacities by Chateauneuf and Jaffray [4]. Gilboa and Schmeidler observed in [18] that the result is true for any capacity and extended it to infinite Ω.

1. Either *there does exist a* $\nu \in \mathcal{V}^B$ *such that for all* $f \in \mathcal{F}$*,*

$$\int_\Omega u(f^*(\omega))\,\nu(d\omega) \geq \int_\Omega u(f(\omega))\,\nu(d\omega); \tag{17}$$

2. or *there is a* $\sigma \in \Delta(\mathcal{F}')$ *such that for all* $A \in \mathcal{N}$

$$\sum_{f \in \mathcal{F}'} \sigma(f)\,[\min_{\omega \in A} u(f(\omega))] > [\min_{\omega \in A} u(f^*(\omega))]. \tag{18}$$

Proof: To apply the result of Proposition 5, let $T = \mathcal{N}$, $S = \mathcal{F}$ and define $U : S \times T \to \mathbf{R}$ by

$$U(s,t) = \min_{\omega \in t} u(s(\omega)).$$

In fact notice that by Eq. (13) and the definition of U, if there is $\alpha \in \Delta(T)$ satisfying (15) then the belief function $\nu \in \mathcal{V}^B$, defined as $\sum_t \alpha(t)\, e_t$, is such that the inequalities (17) are satisfied. ∎

A similar result could be proved for general capacities, but, as we show in Theorem 2, it would really yield a characterization equivalent to the one seen in Corollary 2.

Appendix C Proof of Theorem 2

We only need to prove that if f^* is Choquet rational, then it is Shafer rational. That is, we show that given a capacity ν which rationalizes f^*, i.e., is such that for all $f \in \mathcal{F}$,

$$\int_\Omega f^*(\omega)\,\nu(d\omega) \geq \int_\Omega f(\omega)\,\nu(d\omega), \tag{19}$$

there is a belief function $\mu = \mu(f^*)$ for which (19) holds *mutatis mutandis*. This is done by first constructing the belief function μ as follows: Pick one additive representation $\alpha(\nu)$ of ν, as obtained in Proposition 3. Consider the following correspondence from $\mathcal{E}$ into $\mathcal{N}$:

$$\Lambda(e) \equiv \{A \in \mathcal{B}_e : \bar{f}^*(A) \geq \bar{f}^*(B),\ \forall B \in \mathcal{B}_e\},$$

where the extension of every act in $\mathcal{F}$ to $\mathcal{N}$ is defined in Eq. (4), and let $\lambda : \mathcal{E} \to \mathcal{N}$ be a selection from Λ (i.e., $\lambda(e) \in \Lambda(e)$ for all $e \in \mathcal{E}$). For every $A \in \mathcal{N}$, let $\mathcal{E}(A)$ be defined as follows:

$$\mathcal{E}(A) \equiv \{e \in \mathcal{E} : A = \lambda(e)\}.$$

(The identity of f^* is relevant for that of $\mathcal{E}(A)$. For this reason, the μ we construct depends on f^*.) Consider now the number

$$\varphi(A) = \sum_{e \in \mathcal{E}(A)} \alpha(\nu)(e).$$

It is immediate to see that φ is a function $\varphi : \mathcal{N} \to [0,1]$, and that $\sum_{A\in\mathcal{N}} \varphi(A) = 1$. Let μ be the belief function which has φ as Möbius transform, that is,

$$\mu(A) = \sum_{B\subseteq A} \varphi(B).$$

We now proceed to show that μ rationalizes the choice of f^*. Notice that from Eqs. (4), (19), and (11) we have that for every $f \in \mathcal{F}$,

$$\sum_{e\in\mathcal{E}} \alpha(\nu)(e) \max_{A\in\mathcal{B}_e} [\bar{f}^*(A)] \geq \sum_{e\in\mathcal{E}} \alpha(\nu)(e) \max_{A\in\mathcal{B}_e} [\bar{f}(A)].$$

The left-hand side is rewritten as follows:

$$\sum_{A\in\mathcal{N}} \varphi(A)\, \bar{f}^*(A) = \sum_{e\in\mathcal{E}} \alpha(\nu)(e) \max_{A\in\mathcal{B}_e} [\bar{f}^*(A)].$$

As for the right-hand side, we have that

$$\begin{aligned} \sum_{e\in\mathcal{E}} \alpha(\nu)(e) \max_{A\in\mathcal{B}_e} [\bar{f}(A)] &\geq \sum_{e\in\mathcal{E}} \alpha(\nu)(e)\, [\bar{f}(\lambda(e))] \\ &= \sum_{A\in\mathcal{N}} \varphi(A)\, \bar{f}(A), \end{aligned}$$

where the first inequality follows from the immediate observation that for every $f \in \mathcal{F}$ and $e \in \mathcal{E}$, $\max_{A\in\mathcal{B}_e}[\bar{f}(A)] \geq \bar{f}(\lambda(e))$. We thus conclude that for every $f \in \mathcal{F}$,

$$\sum_{A\in\mathcal{N}} \varphi(A)\, \bar{f}^*(A) \geq \sum_{A\in\mathcal{N}} \varphi(A)\, \bar{f}(A),$$

which, when written as a Choquet integral, gives

$$\int_\Omega f^*(\omega)\, \mu(d\omega) \geq \int_\Omega f(\omega)\, \mu(d\omega).$$

Appendix D Proof of Theorem 3

We start by proving that if an act f is ordinally Shafer dominated, then it cannot be ordinally Shafer rational. For any belief function $\nu \in \mathcal{V}^B$, let $\varphi(\nu)$ be its Möbius transform, and let u be a utility function representing $\succeq$. Let $\mathcal{A} \equiv \{A \in \mathcal{N} : \varphi(\nu)(A) > 0\}$. Since f is ordinally Shafer dominated, there exists $f^* \in \mathcal{F}$ such that $(f^*, A) \succcurlyeq (f, A)$ for all $A \in \mathcal{A}$, with a strict inequality for at least one $A \in \mathcal{A}$. We deduce that

$$\int_\Omega u(f^*(\omega))\, \nu(d\omega) > \int_\Omega u(f(\omega))\, \nu(d\omega).$$

For the other direction, assume that $f^* \in \mathcal{F}$ is not ordinally Shafer dominated. Take any $\mathcal{A} \subseteq \mathcal{N}$. Then by Börgers [3, Lemma], there exists a utility function $\bar{u}$ representing $\succcurlyeq$ and a probability φ on $\mathcal{A}$ such that for all $f \in \mathcal{F}$,

$$\sum_{A \in \mathcal{A}} \varphi(A)\, \bar{u}(f^*, A) \geq \sum_{A \in \mathcal{A}} \varphi(A)\, \bar{u}(f, A).$$

Finally let ν be the belief function induced by the vector of weights φ (see Proposition 4), so that eq. (13) gives the result.

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