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An Imaginary Realistic Market*

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Abstract

An imaginary dimension is added to the market, where imaginary values attach to the price, where the imaginary price creates sequential forces, where the imaginary forces influence market participants' beliefs, where heterogenous beliefs guide the market moves, where the market movements provide images that can or cannot be anticipated by the participants but exist in the reality.

1 Introduction

Whether the mankind fully comprehends the substance of what they create? The market, the actual substance of the economy invented by humans, intrigues various incomprehensible movements to its participants. At the two extremes, capitalism is afraid of the crisis of its economic relations, while communism is afraid of the corruption of its ideological beliefs. Between these two extremes, a mixing sentiment toward these controversial roles widely exits. Even though extensive studies about the market have been done in past decades, there is no sign that these fears would fade from our views in the near future.

To overcome these fears, one needs to trace the root of one's incomprehension on the market. The incomprehension may not be caused by the evolution of the market as the methodology of market analyses evolves simultaneously. It may not be caused by the complication of the market either since the novel indicators become available continuously. Well-developed theories illustrate us the universal laws of supply and demand. Numerous pieces

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of evidence testify their correctness in various conditions. But these results apparently do not inhibit our worries. The market seems to stay restless as if it was always threatened with some kind of unseen damages.

In this paper, I attribute this incomprehension to an incomplete vision of the market character. A market is a process by which the prices of economic elements are established. One may righteously think that the righteousness of the market if it exists, would be reflected entirely through the price, which guides the supply and the demand of goods and chattels, which proposes the amount of the payments. All the actions of the price are done by the standard arithmetic of using real numbers. Despite the obvious fact that the price is observable, this paper shows that the real price should contain an unreal part that is unobservable.

Every real-valued price will be shown to have an unobservable companion that is called the imaginary price. The real-imaginary pair of the price will represent the core pillar of creating the market equilibrium in the superstructure, an excuse for the existence of an equilibrium together with the creation of disequilibrium phenomena. The added imaginary dimension to the equilibrium model can extend the scope of current economic analysis. Also, it can refine some distorted comprehensions. The importance of a market phenomenon no longer arises from the phenomenon itself but from its coordinates in the world of realistic and ideological concepts.

Like the concept of utility, the imaginary price is not yet a concrete measurement. But unlike the utility that emphasizes on the individual preference, the way of the imaginary price associated with the real-valued price makes it an aggregate homogenous impact for all the market participants and meanwhile the same participants can have heterogenous perceptions to this imaginary price as it is invisible for all of them. It is a byproduct generated by the market and can provide a wide range of influences on the market's actions.

The connection between the imaginary price and all sorts of economic actions goes through a market channel of beliefs. While there is no good and no evil imaginary price, there are good and malicious moves. The imaginary price induces heterogenous imaginations with which the receivers form their generic beliefs. Then the beliefs provide the potential sets of actions. The channel of synchronizing the prices and the beliefs is formulated by an invisible hand. The hand maps one's real states to the states that are believed to be. It leads to a coherent system of beliefs, relying upon a few basic assumptions about reality that may or may not have any factual basis. Imaginary prices become ideologies such as coherent and repeated patterns, through the subjective ongoing choices that people make, serving as the seed around which further thought grows.

The market then becomes a visualization of these beliefs. This visualization possesses a hierarchical order as the unobservable imaginary prices are perceived differently in the different position of the market. It morphs the pricing process of demands equaling supplies into a harmonic adaptation of beliefs with which the hierarchical market responses are self-consistent for the observable phenomena. This metamorphic pricing process proposes an alternative interpretation of the market operation. In this process, not only the equilibrium is of interest, but also the disequilibrium and the singularity. The controversial properties of equilibrium and non-equilibrium are consolidated with the help of the imaginary dimension. All these characters will portray an ultimate module of how a market would expand towards its utopian state.

The imaginary perspective intends to give alternative justifications of individual decisions and expectations, to understand the realms of the free market doctrine or even the constitutional limits of the market power. It aims at providing a coherent solution to the disputes between the free and the governed market.

The structure of this paper is as follows. Section 2 starts with specifying the standard characters of the market within a general polynomial economy. Then the necessity of the imaginary dimension, including the imaginary price and the market sentiment, is proposed in Section 3, followed by a concrete way of representing of these imaginary forces in Section 4. In Section 5 and 6, the representation is converted to a hierarchical system of beliefs, by which an alternative aggregation of market elements is established. The new module analyzes the growth of the market in Section 7.

2 Visible Forces in The Market

2.1 Demand and supply

Consider an economy consisting of K sectors where $K \in \mathbb{N}$ is an arbitrary natural number. For any $1 \leq l \leq K$, in the l-th sector, a single type of homogeneous commodities or elements is demanded by $n_l \in \mathbb{N}$ independent heterogenous market participants. The total amount of the demands in the l-th sector is defined by x_l such that

$$x_l^d = \mathbb{E}[X_l^d] = \int X^d(\omega_l) \mathrm{d}\mathbb{P}(\omega_l) = \int \cdots \int \left(\sum_{q=1}^{n_l} X^d(\omega_l^q) \right) \mathrm{d}\mathbb{P}(\omega_l^1) \cdots \mathrm{d}\mathbb{P}(\omega_l^{n_l}) \qquad (1)$$

where $\mathbb{P}(\omega_l^q)$ denotes the probability measure for describing the uncertainty of the endowment ω of the q-th agent spent for buying the l-th element. When there is no need of emphasizing the sector endowment ω_l, the notation X_l^d is used as a shorthand term of $X^d(\omega_l)$, the random sector demand. Besides, the definition of 1 induces that $\mathbb{P}(\omega_l)$, the probability measure of total endowment spent in the l-th sector, satisfies the infinite divisible property such that

$$\mathbb{P}(\omega_l) = \mathbb{P}(\omega_l^1) \star \mathbb{P}(\omega_l^2) \cdots \star \mathbb{P}(\omega_l^{n_l}) \tag{2}$$

for any $n_l \in \mathbb{N}$, where $\star$ stands for the convolution such that

$$\mathbb{P}(\omega_l^1) \star \mathbb{P}(\omega_l^2) = \int \mathbb{P}(\omega_l^1)\mathbb{P}(\omega - \omega_l^1)\mathrm{d}\omega = \int \mathbb{P}(\omega - \omega_l^2)\mathbb{P}(\omega_l^2)\mathrm{d}\omega.$$

The uncertainty of endowments allows the growth and the decline of the individual demands as well as the total demand of the sector. The convoluted probability measure $\mathbb{P}(\omega_l)$ preserves the probabilistic structure under the arbitrary number of participants. Implanting the uncertainty in this economy enables the stochastic growth of demands and allows the evolution of the market size of each sector.

The production rule of this economy follows the combination of available elements across the sectors. To produce a quantity x_l in the l-th sector, the economy can deploy different elements across all sectors $\mathbf{x} = (x_1, \dots, x_K)$. The production function of a producer follows a polynomial of monomials

$$\mathbf{x}^{\mathbf{a}} = x_1^{a_1} \cdot x_2^{a_2} \cdots x_K^{a_K}.$$

The technology profile $\mathbf{a} = (a_1, \dots a_K) \in \mathbb{N}^K$ measures the combinatorial effect of some elements from K sectors. The form of monomials is a generalization of two variables' Cobb–Douglas function. The types of the production $\mathbf{x}^{\mathbf{a}}$ vary across firms regarding $\mathbf{a}$. Let $\mathcal{A}$ be the set of all possible $\mathbf{a} \in \mathcal{A} \subset \mathbb{N}^K$. The size of the set $\mathcal{A}$ is finite. The production function of the l-th sector follows a finite linear combination of $\mathbf{x}^{\mathbf{a}}$ such that

$$x_l = f_l(\mathbf{x}) = \sum_{\mathbf{a} \in \mathcal{A}} c_{l,\mathbf{a}} \mathbf{x}^{\mathbf{a}} = \sum_{(a_1,\dots,a_K) \in \mathcal{A}} c_{l,a_1,\dots a_K} x_1^{a_1} \cdot x_2^{a_2} \cdots x_K^{a_K}, \quad \text{with } c_{l,\mathbf{a}} \in \mathbb{R}. \tag{3}$$

For any sector l, $f_l \in \mathbb{R}[\mathbf{x}]$ where $\mathbb{R}[\mathbf{x}]$ stands for the set of all polynomials that consist of productions of K elements x_1, x_2,..., x_K and that the coefficients $c_{l,\mathbf{a}}$ of the polynomials are in the real number field $\mathbb{R}$. It is known that the set $\mathbb{R}[\mathbf{x}]$ has the same topological structure as the real-valued K-vector space.[1] The term $x_1^{a_1} x_2^{a_2} \cdots x_K^{a_K}$ induces a linear basis of monomials for such a vector space. The coefficients $c_{l,\mathbf{a}}$ measures the relative contribution of each firm to the final output. The choices of $c_{l,\mathbf{a}}$ and the technology profile $\mathbf{a}$ are arbitrary in $\mathbb{R}$ and $\mathbb{N}^K$ respectively.

Definition 1. (Polynomial economy and its equilibria) An exchange economy can produce K types of homogeneous goods. This economy is attached with stochastic growth patterns of demands $\mathbf{x}^d = (x_1^d, \ldots, x_K^d)$ defined in (1) and with polynomial productions $\mathbf{f} = (f_1, \ldots, f_K)$ defined as (3) in $\mathbb{R}[\mathbf{x}]$. The pair $(\mathbf{f}, \mathbf{x}^d)$ is called the polynomial economy. The equilibria set of this economy is given by

$$\mathbb{V}(\mathbf{f}, \mathbf{x}^d) = \left\{ \mathbf{x} = (x_1, \ldots, x_K) \in \mathbb{R}^K : \ f_l(x_1, \ldots, x_K) - x_l^d = 0 \text{ for } 1 \leq l \leq K \right\}.$$

The set $\mathbb{V}(\mathbf{f}, \mathbf{x}^d)$ is about all the solutions of the simultaneous system of polynomial equations $\mathbf{f}(\mathbf{x}) - \mathbf{x}^d = 0$.

The equilibria of such an exchange economy are defined by a class of polynomial production functions $\mathbf{f} \in \mathbb{R}[\mathbf{x}]$ that meet the total demands $\mathbf{f}(\mathbf{x}) = \mathbf{x}^d$. Thus the sets of roots for the polynomial systems of equations are also characterized by $c_{l,\mathbf{a}}$ and $\mathbf{a}$, both as the parameters of the polynomials $\mathbf{f}$, and by $\mathbf{x}^d$. Given $\mathbf{f}$ and $\mathbf{x}^d$, the set of $\mathbf{x}$ satisfying $\mathbf{f}(\mathbf{x}) = \mathbf{x}^d$ is called the algebraic variety in algebraic geometry. The set of equilibria as the set of roots is an algebraic variety. The set $\mathbb{V}(\mathbf{f}, \mathbf{x}^d)$ in Definition 1 refers to the affine variety.

One example of the polynomial economy is the Leontief's input-output model

$$\mathbf{x} = \mathbf{f}(\mathbf{x}) = \mathbf{C}\mathbf{x}$$

where $\mathbf{C}$ is the input–output matrix and any c_{ij} in $\mathbf{C}$ means that in order to produce one unit in the j-th sector, the economy must use c_{ij} units from sector i. The set of equilibria of this model corresponds to linear variety, a special case of the affine variety.

Describing the exchange economies by the polynomial system was first proposed in [9]. In [9], the variety result is applied to the correspondence of preferences, then the algebraic geometry setting characterizes the equilibria of exchange economies. The preference and the utility are not in use here. Definition 1 suggests a specification of equilibria based solely on

[1] In mathematics, $\mathbb{F}[\mathbf{x}]$ is a ring called polynomial ring for any field $\mathbb{F}$ such as $\mathbb{F} = \mathbb{Q}$ (rational number field) or $\mathbb{F} = \mathbb{R}$ (real number field) or $\mathbb{F} = \mathbb{C}$ (complex number field). One can see that under addition and multiplication, $\mathbb{F}[\mathbf{x}]$ satisfies all of the field axioms except for the existence of multiplicative inverses.

the supply and the demand. The endowments, which may potentially relate to preferences or utilities, are described by stochastic laws and are not explicitly used for the equilibria.

2.2 Price

The market of an economy is considered as a system in which the allocations of goods and services are fulfilled by interactions amongst the participants. In addition, any participants who intend to trade their elements must label the prices of their products. The introduction of price particularly enhances the allocation processes. Because the price gives the signals of the market exceeded supply or demand, it is believed that in the free market economy the pricing system is an efficient and ideal way of clearing the market. When the commodities are labeled by their prices, the equilibrium allocations in the previous exchange economy are supposed to stay as equilibria. The following claim is to specify the function of introducing the price to a free market.

Claim. (Price in a free market) (i) A price quotes each unit or quantity of the economic elements in the free market. (ii) In the free market, the market clear is only conducted by the pricing system.

The above claim is to summarize the spirit of the free market pricing system. The first part of the claim is obvious because any economic element found in the market goes with its labelled price. The second part of the claim may be less obvious. One deductive reasoning is given as follows. Price reflects interaction between supply and demand. In a free market economy, apart from price, there is no other quantifier of indicating the supply and demand relation. Thus the strategy of clearing the market in such an economy must go with the price.

This claim of price, however, conceals some essential puzzles. To reveal these puzzles one needs to distinguish the function of pricing and the function of clearing the market. The following definitions quantify the price and market clear strategy as two separated classes of functions. To accommodate the previous polynomial economy and its equilibria, the pricing functions and market clear strategies are both assumed to be in the set of $\mathbb{R}[\mathbf{x}]$. That is, they are from the set of polynomial functions.

Definition 2. (Prices of polynomial market) For the polynomial economy $(\mathbf{f}, \mathbf{x}^d)$, the price vector $\mathbf{h}$ is set to equate the quantity of supply and demand

$$\left\langle (\mathbf{f}, \mathbf{x}^d) \right\rangle_{\mathbb{R}} = \left\{ \mathbf{h} = (h_1, \dots, h_K) \in \mathbb{R}^K[\mathbf{x}] : \sum_{l=1}^{K} h_l(\mathbf{x}) f_l(\mathbf{x}) = \sum_{l=1}^{K} h_l(\mathbf{x}) x_l^d. \right\}.$$

As the price, in reality, is attached to the unit or the quantity of the elements, there is an explicit expression of linear products $h_l(\mathbf{x})f_l(\mathbf{x})$ and $h_l(\mathbf{x})x_l^d$ for calculating the total amounts of prices on the supply side and the demand side. Note that in contrary to the existing settings, price here is not assumed to be bounded or to be convexified. This is more realistic. As the number of fiat currencies is consistently expanding, it is unnecessary to consider the bounded prices whose present measurements are the fiat currencies.

Definition 3. (Strategies of clearing polynomial market) For the equilibria set of the exchange economy $\mathbb{V}(\mathbf{f},\mathbf{x}^d) \subset \mathbb{R}^K$ in Definition 1, the market clear function belongs to the set

$$\mathbb{I}_{\mathbb{R}}(\mathbb{V}) = \left\{ g \in \mathbb{R}[\mathbf{x}] : g(f_1(\mathbf{x}) - x_1^d, \ldots, f_K(\mathbf{x}) - x_K^d) = 0, \text{ for all } \mathbf{x} \in \mathbb{V}(\mathbf{f},\mathbf{x}^d) \right\}.$$

Definition 3 implies that the market clear function does not distort the equilibrium set obtained in the pure exchange economy. It is straightforward to see that in the polynomial economy $(\mathbf{f},\mathbf{x}^d)$, the price function satisfies the market clear strategy

$$g(f_1(\mathbf{x}) - x_1^d, \ldots, f_K(\mathbf{x}) - x_K^d) = \sum_{l=1}^{k} h_l(\mathbf{x})(f_l(\mathbf{x}) - x_l^d) = 0.$$

Thus $\left\langle (\mathbf{f},\mathbf{x}^d) \right\rangle_{\mathbb{R}} \subset \mathbb{I}_{\mathbb{R}}(\mathbb{V})$.[2] Unlike the price function that linearly multiplies with the quantity of the production, the market clear strategy is an aggregated function with respect to all the sectors simultaneously. From the claim of price, any market clear outcome in $\mathbb{I}_{\mathbb{R}}(\mathbb{V})$ must be conducted by the pricing system. According to this claim, even though a strategy may consider to clear the aggregate market excess $(f_1(\mathbf{x}) - x_1^d, \ldots, f_K(\mathbf{x}) - x_K^d)$, this strategy is finally implemented by the pricing system over the individual type of the quantity $(f_l(\mathbf{x}) - x_l^d)$ for $1 \leq l \leq K$. In short, if the claim is true, one would expect $\mathbb{I}_{\mathbb{R}}(\mathbb{V}) = \left\langle (\mathbf{f},\mathbf{x}^d) \right\rangle_{\mathbb{R}}$ because there is no other instrument to clear the market. Unfortunately, Definition 2 of the price can not guarantee $\mathbb{I}_{\mathbb{R}}(\mathbb{V}) = \left\langle (\mathbf{f},\mathbf{x}^d) \right\rangle_{\mathbb{R}}$. Such a deficiency leads to a further investigation of the character of the price.

[2] The set $\mathbb{I}_{\mathbb{R}}(\mathbb{V})$ is a subset of $\mathbb{R}[\mathbf{x}]$ and it is called an ideal when $\mathbb{V}$ is an affine variety. This result is given in [2, lemma 4 in Ch 1.4].

3 Invisible Forces in The Market

Note that the exchange polynomial economy given in Definition 1 is restricted to $\mathbb{R}[\mathbf{x}]$, the set of polynomials with real coefficients. Although the real number field $\mathbb{R}$ is practically meaningful, one cannot always find the roots of $f_l(\mathbf{x}) = x_l^d$ on the field $\mathbb{R}$ because $\mathbb{R}$ is not algebraically closed. Consider an exchange economy with two elements

$$(\frac{1}{2}x_1 + x_2) - 9 = 0, \ (x_1^2 + 10) - 6 = 0. \tag{4}$$

where $f_1(x_1, x_2) = \frac{1}{2}x_1 + x_2$, $f_2(x_1) = x_1^2 + 10$, $x_1^d = 9$ and $x_2^d = 6$. It is easy to see that the solutions depend on $x_1 = \sqrt{-4}$ which is an imaginary number due to $\sqrt{-4} = 2\mathrm{i} \in \mathbb{C}$ and $\mathrm{i}^2 = -1$. Thus $\mathbb{V}_{\mathbb{R}}(f_1, f_2, 9, 6)$ has no solution in $\mathbb{R}$. However, the fundamental theorem of algebra states that any non-constant polynomial $f \in \mathbb{C}[\mathbf{x}]$ has a root in the complex number filed $\mathbb{C}$. Thus the field $\mathbb{C}$ is algebraically closed.

The lack of algebraical closedness for $\mathbb{R}[\mathbf{x}]$ is harmless when the attention is restricted solely to the exchange economy. Because the feasible equilibrium allocation in an exchange economy should always be some real quantity that is observable and available for measurements. Furthermore, there is no invisible force for creating an opportunity of trading infeasible quantities such as 2i. However, once the pricing system is implemented, the strategy of clearing the market would generate invisible forces that can invoke this deficiency and can make use of imaginary units to fulfill the necessary execution.

3.1 Imaginary price

The price $h_l(\mathbf{x})$ is universally accepted as a real-valued function for its observable nature. On the other hand, this function of price is endowed by the market, and the market consists not only the real figures but also the imaginary creatures. These imaginary creatures conceptualize abstract properties, spread spiritual and psychological influences, create cultural parasites. They adhere to the economic elements in the market and reveal their values in the imaginary dimension. One imaginary property may be subjective. However, the market collects these subjective creeds and forms an invisible force. This force will price the commodities, but the value generated by this imaginary force is incompatible to the real-valued price. The following theorem concretizes the role of imaginary price. The relation between imaginary number and the sentiment of the market will be discussed in Section 3.2.

Theorem 1. *(Fundamental theorem of imaginary price) Let the exchange polynomial economy* $(\mathbf{f}, \mathbf{x}^d)$ *as in Definition 1 whose equilibrium set* $\mathbb{V}(\mathbf{f}, \mathbf{x}^d)$ *is not empty. Extend the class* $\mathbb{R}[\mathbf{x}]$ *of the polynomial price function* $\mathbf{h}(\mathbf{x})$ *and the market clear function* $g(\mathbf{f}(\mathbf{x}) - \mathbf{x}^d)$ *in Definition 2 and 3 to* $\mathbb{C}[\mathbf{x}]$*, namely a set of polynomials with complex coefficients. Then*

$$\left\langle (\mathbf{f}, \mathbf{x}^d) \right\rangle_{\mathbb{C}} = \mathbb{I}_{\mathbb{C}}(\mathbb{V}).$$

In particular, for certain integer number m*, acting* m *times market clear strategy* g *is equivalent to clear the market by the price function*

$$\sum_{l=1}^{k} h_l(\mathbf{x})(f_l(\mathbf{x}) - x_l) = g^m(f_1 - x_1, \ldots, f_k - x_k) = 0$$

for $h_1, \ldots, h_k \in \mathbb{C}[\mathbf{x}]$*. Let* $\mathbf{h}(\mathbf{x}) = (h_1(\mathbf{x}), \ldots, h_K(\mathbf{x}))$ *and*

$$\mathbf{h}(\mathbf{x}) = \mathbf{p}(\mathbf{x}) + \mathrm{i}\mathbf{y}(\mathbf{x})$$

composes of $\mathbf{p}(\mathbf{x}), \mathbf{y}(\mathbf{x}) \in \mathbb{R}^K[\mathbf{x}]$*. The imaginary part* $\mathbf{y}(\mathbf{x})$ *is called the imaginary price.*

The theorem resolves the previous puzzle $\left\langle (\mathbf{f}, \mathbf{x}^d) \right\rangle_{\mathbb{R}} \neq \mathbb{I}_{\mathbb{R}}(\mathbb{V})$. That is, the strategy function of clearing the market is incompatible with the price function. By introducing the imaginary number field for the price function, one extends the price function from $\left\langle (\mathbf{f}, \mathbf{x}^d) \right\rangle_{\mathbb{R}}$ to $\left\langle (\mathbf{f}, \mathbf{x}^d) \right\rangle_{\mathbb{C}}$, an algebraically closed field extension. In this pricing system, every market clear strategy is compatible with a pricing function in $\left\langle (\mathbf{f}, \mathbf{x}^d) \right\rangle_{\mathbb{C}}$.

As the imaginary price extends the equilibrium set, an infeasible allocation in $\mathbb{I}_{\mathbb{R}}(\mathbb{V})$ may become feasible in $\mathbb{I}_{\mathbb{C}}(\mathbb{V}) = \left\langle (\mathbf{f}, \mathbf{x}^d) \right\rangle_{\mathbb{C}}$. For example, $x_1 = \sqrt{-4}$ in (4) is infeasible as it violates the rule of the production where the economic element is real. Suppose only the real-valued price $p(x_1, x_2)$ is given to this element, then

$$p(x_1, x_2)(x_1 - \sqrt{-4}) = 0.$$

This allocation is still infeasible because $\sqrt{-4} \notin \mathbb{V}(\mathbf{f}, \mathbf{x}^d) \subset \mathbb{R}$ in the exchange economy. When the price function contains an imaginary part, for example

$$h(x_1, x_2) = p(x_1, x_2)(x_1^2 - 4)(x_1 + \sqrt{-4}) \in \mathbb{C}[x_1, x_2],$$

$x_1 = \sqrt{-4}$ becomes feasible in

$$h(x_1, x_2)(x_1 - \sqrt{-4}) = p(x_1, x_2)(x_1^4 - 16) = 0.$$

By adequately pricing the imaginary term $(x_1 - \sqrt{-4})$, the system lists a real-value price $p(x_1, x_2)$ for $(x_1^4 - 16)$ that is feasible for the production. The term $(x_1 + \sqrt{-4})$ in the pricing function contributes a compensation for the imaginary quantity $\sqrt{-4}$.

Also, Theorem 1 shows that any l-th element $x_l \in \mathbb{R}$ priced by the real-valued function $p_l(\mathbf{x})$ is not affected by the imaginary price $\mathrm{i}y_l(\mathbf{x})$ because they locate in different dimensions

$$h_l(\mathbf{x})x_l = p_l(\mathbf{x})x_l + \mathrm{i}y_l(\mathbf{x})x_l.$$

The imaginary priced quantity $y_l(\mathbf{x})x_l$ only appears in the imaginary dimension. Thus for real-valued measurements in the market, the imaginary price $\mathrm{i}y_l(\mathbf{x})$ does not matter much in either before or after the exchanges. However, the individual's subjective sentiment invoked by $\mathrm{i}y_l(\mathbf{x})$ may bring in some real effects caused by a collection of these imaginary sentiments. Real market action may not be revealed by a real-valued price $p_l(\mathbf{x})$ but by a real price $h_l(\mathbf{x})$ that contains some imagination.

3.2 Market sentiment

By the definition, the imaginary price enters the imaginary dimension and is directly unmeasurable regarding the real-valued price. However, due to Theorem 1, the price now is known as an aggregated function of the market. The aggregation can form additional effects. The real effect of imaginary price is assumed to propagate through the market sentiment. I consider two types of opposite or complementary sentiment as there are only two possibilities for the sentiment of the market, positive (bullish) and negative (bearish) sentiment. Such a dual system is firmly rooted in a rich set of literature, from ancient eastern and western philosophy works to modern physics, economics or social models.

The market price, as $h = p + \mathrm{i}y$ in the current setting, composes of numeric values $p, y \in \mathbb{R}$. The imaginary price will drive the sentiment. As the market sentiment is an intrinsic and invisible character that accompanies with human's intuition towards the changes, it can be represented by a pair of hypothetical binary indicators in an interactive context.

Consider a minimax game of these two types of sentiment Yin 阴(z) and Yang 阳(z) with

$z \in \mathbb{R}$:

$$\min_{阴} \max_{阳} 阴(z) \times 阳(z), \quad 阴(0) = 阳(0) = \text{initial condition.}$$

The Yin tends to minimize the product value while the Yang tends to maximize it. The following systems of differential equations describe the gradient descent of the minimax game

$$\begin{cases} \frac{\partial 阴(z)}{\partial z} = & -阳(z), \\ \frac{\partial 阳(z)}{\partial z} = & 阴(z), \end{cases} \quad \text{which implies } (*) \begin{cases} \frac{\partial^2 阳(z)}{\partial z^2} = & -阳(z), \\ \frac{\partial^2 阳(z)}{\partial z^2} = & \frac{\partial 阴(z)}{\partial z}. \end{cases}$$

These are standard homogenous second order differential equations. Differential equations of the $(*)$ form have sinusoidals as their set of basis functions of solutions. With the initial condition, the unique solution of $(*)$ is

$$\begin{pmatrix} 阳(z) \\ 阴(z) \end{pmatrix} = \underbrace{\begin{pmatrix} \cos z & -\sin z \\ \sin z & \cos z \end{pmatrix}}_{\mathbf{R}} \begin{pmatrix} 阳(0) \\ 阴(0) \end{pmatrix} \tag{5}$$

where the matrix $\mathbf{R}$ forms a cyclic operator for the dynamical interaction between Yin and Yang. This cyclic operator characterizes the dynamical interaction between Yin and Yang. A yin and a yang have no end, the end of the day begins at the beginning. Since the Yin and Yang are invisible, their initial condition is less critical than the cyclic operator. The following proposition connects this matrix operator of the invisible market sentiment with the imaginary priced elements.

Proposition 1. *Given the K-vector of random elements for the demands*

$$\mathbf{X}^d = (X^d(\omega_1), \ldots, X^d(\omega_K))$$

defined in (1) and the price vector

$$\mathbf{h} = \mathbf{h}(\mathbf{x}) = \mathbf{p}(\mathbf{x}) + \mathrm{i}\mathbf{y}(\mathbf{x}),$$

the cyclic operator $\mathbf{R}$ can embed the imaginary quantity $\mathrm{i}\mathbf{y}^\top \mathbf{X}^d \in \mathbb{C}$ due to the invisible interaction of (5) generated by the market sentiment. The explicit relation can be expressed compactly by

$$e^{\mathrm{i}\mathbf{y}^\top \mathbf{X}^d} = \cos \mathbf{y}^\top \mathbf{X}^d + \mathrm{i} \sin \mathbf{y}^\top \mathbf{X}^d.$$

The equality from Proposition 1 is nothing else but the Euler's formula on the imaginary pricing quantity. Although $\mathrm{i}\mathbf{y}^\top\mathbf{X}^d$ is imaginary and thus it is invisible, $e^{\mathrm{i}\mathbf{y}^\top\mathbf{X}^d}$ has a real-valued term $\cos\mathbf{y}^\top\mathbf{X}^d$. The exponential function induced by the interactions of the dual sentiments generates a visible cyclic pattern when the real-valued cosine function continuously changes with respect to $\mathbf{y}^\top\mathbf{X}^d$. This seemingly contradicting behavior of the imaginary price discloses a new perspective for quantifying the movement of the market.

Proposition 1 uses only the random quantities $\mathbf{X}^d$ from the demand side because, in the following sections, the effects of imaginary forces will be first derived from the demand side where the uncertainty of the endowments can generate heterogenous perceptions for the imaginary quantities. The supply side would receive these imaginary effects sequentially through the market interactions on the real-valued quantities. This approach is different from the standard equilibrium analysis of the price. The reason is that the imaginary price, unlike the real-valued price, cannot be simultaneously observed from both the demand and the supply side. Also, the decisions regarding individuals, sectors or the whole market may construct somewhat different views by the imaginary dimension. Thus it is better to start the analysis on one side of the market. Since the infinite divisible property from the total demand defined in (1) can disentangle the individual uncertainty law from the aggregated one, it is natural to start with the demand side.

4 Invisible Hands and Harmonic Market Powers

So far the invisible forces, the imaginary price $\mathrm{i}\mathbf{y}$ and the interactions of dual sentiment $e^{\mathrm{i}\mathbf{y}^\top\mathbf{X}^d}$, all associate to the result of extending pricing function to the complex number field. Nevertheless, the invisible behaviors of the market have been known as a commonly shared creed since the rise of the market economy. The representative works, among many others, include the studies of Protestant ethic and the spirit of capitalism by Max Weber and of the theory of moral sentiments by Adam Smith. In particular, the compact term, the invisible hands, has been widely accepted to describe an invisible force of the market.

In the current context, by mapping the imaginary price $\mathrm{i}\mathbf{y}$ to $e^{\mathrm{i}\mathbf{y}^\top\mathbf{X}^d}$, the exponential function is capable of visualizing a movement generated by the imaginary price. The cyclic function $\cos\mathbf{y}^\top\mathbf{X}^d$ like the price function is a collective movement of the market. It is formed by the creeds of the participants. Participants create a common norm when they interact in the market and observe the outcome of their interactions. Because this movement is originated not in isolated individuals alone and all the participants in the market commonly share its vision, I consider the invisible hands drive this kind of moves. The "hands" in

Proposition 1 are represented as an exponential function.

The sole assumption of "invisible hand" is that it amplifies the information during its transmission. Consider the dual types of market sentiment. When one sentiment intrigues an infinitesimal change of some market quantity, the order of the change magnitude is proportionally perceived by the other group holding the opposite vision towards the movement. For a small amount of the change, such a propagation should not matter so that participants in the market hardly recognize the function of these "hands". The following definition characterizes the invisible hands in terms of exponential maps.

Definition 4. (Invisible hand) An invisible hand is a continuous map $I : \mathbb{C} \to \mathbb{C}$ for $v \in \mathbb{C}$ such that

$$I(v-\delta) = I(v) - \delta I(v),\; I(v-\mathrm{i}\delta) = I(v) - \mathrm{i}\delta I(v) \tag{6}$$

where $\delta \in \mathbb{R}$ is an infinitesimal and $I(0) = 1$.

Note that (6) characterizes the amplification. An infinitesimal change δ of v is amplified by $I(\cdot)$ so that $I(v-\delta)$ is proportional to $I(v)(1-\delta)$. The amplification also holds in the imaginary dimension so that $I(v-\mathrm{i}\delta)$ is equivalent to $I(v)(1-\mathrm{i}\delta)$. Another interpretation of (6) is that under the invisible hand, $\delta I(v)$ compensates the lost $I(v-\delta)$ from $I(v)$. A little algebra gives the expression $I(\cdot)$ of (6) as an exponential function. Definition 4 can be expressed as

$$\lim_{\delta\to 0}\frac{e^{v}-e^{(v-\delta)}}{\delta} = \frac{\mathrm{d}e^{v}}{\mathrm{d}v} = e^{v},\; \lim_{\delta\to 0}\frac{e^{\mathrm{i}v}-e^{(\mathrm{i}v-\mathrm{i}\delta)}}{\delta} = \frac{\mathrm{d}e^{\mathrm{i}v}}{\mathrm{d}v} = \mathrm{i}e^{v}.$$

Hence the invisible hand in the current context is $I(\cdot) = \exp(\cdot)$. When δ is an infinitesimal, it is known that the map e^{δ} is unrecognizable as $e^{v+\delta} \approx e^{v}$ when $e^{\delta} \approx 1$. The exponentiation propagates the market quantity but it hardly twists an individual quantity if it is small.

In addition, this exponential map is an invariant map for both individuals and groups. Suppose the demand $X^d(\omega_l^q)$ of an individual participant q in the l-th sector is mapped onto $e^{-h_l X^d(\omega_l^q)}$ given the price $h_l = h_l(\mathbf{x})$ and the total market production $\mathbf{x}$. The exponentiation of the total demand X_l^d in the l-th sector is composed by the individual exponents

$$\exp\left\{-h_l X_l^d\right\} = \exp\left\{-\sum_{q=1}^{n_l} X^d(\omega_l^q)\right\} = \exp\left\{-h_l X^d(\omega_l^1)\right\}\cdots\exp\left\{-h_l X^d(\omega_l^{n_l})\right\}.$$

The power of the market demand exponents follows

$$e^{-h_1 X_1^d}\cdots e^{-h_K X_K^d} = e^{-\mathbf{h}^\top \mathbf{X}^d}. \tag{7}$$

Similarly, the supply side has the map $e^{\mathbf{h}^\top \mathbf{f}(\mathbf{x})}$. Thus one can find the invisible hands are functioning indifferently across the individuals and the aggregates. The market exponents derived from Definition 4 are the market quantities from both the supply and the demand sides. The market powers of demands $e^{-\mathbf{h}^\top \mathbf{X}^d}$ and supplies $e^{\mathbf{h}^\top \mathbf{f}(\mathbf{x})}$ are exponentially mapped from the algebraic polynomials to the infinite differentiable sets meanwhile they preserve the Euclidean structure locally.[3]

The behavior of the exponential function attracts a wide range of attention as it is used to describe the steady growth of populations, of numbers of things, of bank accounts with compound interest rates, etc. One can think of these compounding effects are incepted the individual, and then they are expanded to sectors, and finally to the whole market. Thus the exponential map or the invisible hand can amplify individual visions towards the market values and can create a global vision by collecting individuals'. The global vision, as market power, has some interesting properties.

Theorem 2. *(Harmonic Market Power) Given the demand $\mathbf{X}^d$, the power of the market demand exponent $e^{-\mathbf{h}^\top \mathbf{X}^d}$ in (7) induces*

$$e^{-\mathbf{h}^\top \mathbf{X}^d} = u(\mathbf{p}, \mathbf{y}) + \mathrm{i}v(\mathbf{p}, \mathbf{y})$$

where $e^{-\mathbf{p}^\top \mathbf{X}^d} \cos(-\mathbf{y}^\top \mathbf{X}^d) = u(\mathbf{p}, \mathbf{y})$ and $e^{-\mathbf{p}^\top \mathbf{X}^d} \sin(-\mathbf{y}^\top \mathbf{X}^d) = v(\mathbf{p}, \mathbf{y})$. The marginal market power of demands satisfies the conservation law such that

$$\frac{\partial u(\mathbf{p}, \mathbf{y})}{\partial \mathbf{p}} = \frac{\partial v(\mathbf{p}, \mathbf{y})}{\partial \mathbf{y}}, \quad \frac{\partial u(\mathbf{p}, \mathbf{y})}{\partial \mathbf{y}} = -\frac{\partial v(\mathbf{p}, \mathbf{y})}{\partial \mathbf{p}} \tag{8}$$

In other words, the real-valued price and the imaginary price create a harmonic power under the invisible hands. Given the supply $\mathbf{x}$, similar results hold for the power of the market supply exponent.

Theorem 2 implies that when market power deviates from the equilibrium, the marginal power for the imaginary part $v(\mathbf{p}, \mathbf{y})$ will compensate the marginal power for the real-valued one, and vice versa. Then by observing the real change, one can infer the imaginary one. Take an example of $e^{-\mathbf{h}^\top \mathbf{X}^d}$. When the demand $\mathbf{X}^d$ is fixed, and the price function $\mathbf{h} = \mathbf{h}(\mathbf{x})$ depends on the total production $\mathbf{x}$, the change of $\mathbf{x}$ induces the diversity of real-valued price

[3]This property is known as forming a manifold in geometry. Generally speaking, the exponential function of a polynomial ring $\mathbb{C}[\mathbf{x}]$ gives an exponential ring or a subset in an exponential field. If it gives an exponential field, then the map is a homomorphism between an additive algebraic structure and a multiplicative group structure.

$\mathbf{p}$ and imaginary price $\mathbf{y}$ simultaneously. However, the price pair $(\mathbf{p}, \mathbf{y})$ is disequilibrium because the production $\mathbf{x}$ now must be different from the demand $\mathbf{X}^d$. For varying $\mathbf{x}$ in $\mathbf{h} = \mathbf{h}(\mathbf{x})$, the expression of (8) is

$$\frac{\partial u(\mathbf{p},\mathbf{y})}{\partial p_l}\frac{\mathrm{d}p_l(\mathbf{x})}{\mathrm{d}x_l} = \frac{\partial v(\mathbf{p},\mathbf{y})}{\partial y_l}\frac{\mathrm{d}y_l(\mathbf{x})}{\mathrm{d}x_l}, \quad \frac{\partial v(\mathbf{p},\mathbf{y})}{\partial p_l}\frac{\mathrm{d}p_l(\mathbf{x})}{\mathrm{d}x_l} = -\frac{\partial u(\mathbf{p},\mathbf{y})}{\partial y_l}\frac{\mathrm{d}y_l(\mathbf{x})}{\mathrm{d}x_l}.$$

It is clear that the ratio of the derivatives of real-valued and imaginary price is a constant such that

$$\begin{aligned}\left(\frac{\mathrm{d}p_l(\mathbf{x})}{\mathrm{d}x_l}\right)/\left(\frac{\mathrm{d}y_l(\mathbf{x})}{\mathrm{d}x_l}\right) &= \left(\frac{\partial v(\mathbf{p},\mathbf{y})}{\partial y_l}\right)/\left(\frac{\partial u(\mathbf{p},\mathbf{y})}{\partial p_l}\right) \\ &\overset{(*)}{=} -\left(\frac{\partial u(\mathbf{p},\mathbf{y})}{\partial y_l}\right)/\left(\frac{\partial v(\mathbf{p},\mathbf{y})}{\partial p_l}\right)\end{aligned}$$

for any $1 \le l \le K$. Then the equality $\overset{(*)}{=}$ implies a type of conservation of market power for the real and imaginary terms

$$\sum_{l=1}^{K}\left[\frac{\partial v(\mathbf{p},\mathbf{y})}{\partial y_l}\frac{\partial v(\mathbf{p},\mathbf{y})}{\partial p_l} + \frac{\partial u(\mathbf{p},\mathbf{y})}{\partial y_l}\frac{\partial u(\mathbf{p},\mathbf{y})}{\partial p_l}\right] = 0.$$

The derivative terms of $u(\mathbf{p}, \mathbf{y})$ and $v(\mathbf{p}, \mathbf{y})$ in (8) characterize how the harmonic power acts into the market.

Harmonicity of (8) gives a criterion to measure the possible distortion in the imaginary dimension by using the real-valued result. The harmonic property of the market power extends the scope of the dual sentiment. While the dual sentiment only holds for the imaginary price, the harmonic property works for the whole price function $\mathbf{h}$. The origination for the harmonic market power, like the dual sentiment, can be attributed to the suspicion about a sequence of sustainable growths or declines. For example, a modest percentage growth may equate to huge escalations. An investor earning a constant annual return on their investment would find the capital doubling within a few years. But the same exponential power, so advantageous to investors, may lead to a potential Malthusian catastrophe in population growth. Therefore, the market maintains the harmonic towards the disequilibrium movements through balancing the real and imaginary parts of the market power.

5 Beliefs of the Market Power

The invisible hands and the dual sentiment both make the imaginary price indirectly affect the observed market outcomes. For example, the real-valued power of the market demand $e^{-\mathbf{h}^{\top}\mathbf{X}^{d}}$ is $\cos(-\mathbf{y}^{\top}\mathbf{X}^{d})e^{-\mathbf{p}^{\top}\mathbf{X}^{d}}$ that contains the cyclic part generated by the imaginary pricing demand $\mathbf{y}^{\top}\mathbf{X}^{d}$. Although the power $e^{-\mathbf{h}^{\top}\mathbf{X}^{d}}$ is a conceptual definition, this real-valued power $\cos(-\mathbf{y}^{\top}\mathbf{X}^{d})e^{-\mathbf{p}^{\top}\mathbf{X}^{d}}$ can be a potential force of initiating realistic further movements in the market.

A movement that is initiated by the participants may be caused by the beliefs of their owned powers in the market. That is, beliefs of the market power can be incepted into the market actions of these participants. Invisible hands implant the inceptions. Then the participants construct their perspectives about these powers. These perspectives will become the beliefs of participants. Participants act, react and interact in the market under the guided beliefs. This is a process of intriguing the market movement by just incepting one's conceptual power.

Participants in different level of the market possess different level of powers. The hierarchical structure of the power of demands can be viewed as follows:

Level	Real-Valued Power	Perceived Power
(Individual)	$\cos(-y_l X^d(\omega_l^q)) \exp\left\{-p_l X^d(\omega_l^q)\right\}$	$\exp\left\{-p_l X^d(\omega_l^q)\right\}$
	$\Downarrow$	
(Sector)	$\cos(-y_l X_l^d) \exp\left\{-p_l \sum_{q=1}^{n_l} X^d(\omega_l^q)\right\}$	$\exp\left\{-p_l X_l^d\right\}$
	$\Downarrow$	
(Market)	$\cos(-\mathbf{y}^{\top}\mathbf{X}^{d}) \exp\left\{-\sum_{l=1}^{K}\sum_{q=1}^{n_l} p_l X^d(\omega_l^q)\right\}$	$e^{-\mathbf{p}^{\top}\mathbf{X}^{d}}$

For an individual q in the l-th sector, the real-valued power is the real part of $\exp\left\{-h_l X^d(\omega_l^q)\right\}$ that is created by the invisible hand. Nevertheless individual cannot fully perceive this power since the imaginary price $y_l = y_l(\mathbf{x})$ in $\cos(-y_l X^d(\omega_l^q))$, a market level information is unavailable to an individual. The individual can construct his perspective of this power using only the real-valued price. Hence, the perceived power of this individual is $\exp\left\{-p_l X^d(\omega_l^q)\right\}$. A similar argument is applicable to the power at the sector level. However, at the market level the power has a different implication. Because if the imaginary price can be measured then the planner in principle can construct a correct perception of the power. Thus, it is better to consider these two cases separately. Section 5.1 will consider the implication of the perceived power to individual beliefs. Section 5.2 will discuss the collective belief of individuals' for the sector. The market level collective belief will be explained in Section 6.

5.1 Individual beliefs

Suppose that an individual obtains the perceived power $\exp\left\{-p_l X^d(\omega_l^q)\right\}$. The uncertain endowment induces the randomness of $X(\omega_l^q)$ that is known to follow the distribution $\mathbb{P}(\omega_l^q)$. The expected power of this individual is given by

$$\begin{aligned}\mathbb{E}\left[e^{-p_l X^d(\omega_l^q)}\right] &= \int e^{-p_l X^d(\omega_l^q)} \mathrm{d}\mathbb{P}(\omega_l^q) \\ &= \exp\left\{\int_0^\infty \left[\frac{(1-e^{-p_l X^d(\omega_l^q)})}{X^d(\omega_l^q)}\right] \mathrm{d}\mathbb{P}(\omega_l^q)\right\} = e^{-\psi_q(p_l)}\end{aligned} \tag{9}$$

The equation of (9) is nothing else but the Laplace transform of $\mathbb{P}(\omega_l^q)$. The proof of the last representation in (9) is given in Appendix A.4. The exponential form $e^{-\psi_q(p_l)}$ comes from the infinite divisible property of $\mathbb{P}(\omega_l^q)$. The expected power takes into account the uncertainty of endowments. One can adjust the market power by merely reweighing the expected power. This adjustment leads to the definition of individual beliefs.

Definition 5. (Individual beliefs) Given the price p_l, the belief $\pi(\omega_l^q)$ gives the probability of the demand power under current endowment ω_l^q over the expected demand power. That is

$$\pi(\omega_l^q) = \frac{e^{-p_l X^d(\omega_l^q)}}{\mathbb{E}\left[e^{-p_l X^d(\omega_l^q)}\right]} = \exp\left\{-p_l X^d(\omega_l^q) + \psi_q(p_l)\right\} \tag{10}$$

where $\psi_q(p_l)$ is given in (9).

The belief $\pi(\omega_l^q)$ is a probability function as it is a non-negative function and it satisfies

$$\mathbb{E}\left[\pi(\omega_l^q)\right] = \mathbb{E}\left[\frac{e^{-p_l X^d(\omega_l^q)}}{\mathbb{E}\left[e^{-p_l X^d(\omega_l^q)}\right]}\right] = 1.$$

This probabilistic belief is different from the probabilistic law of the uncertain endowment $\mathbb{P}(\omega_l^q)$. The former is guided by the information of the price of the demand thus it varies with p_l, while the later is intrinsic in the system. In other words, the probabilistic law $\mathbb{P}(\omega_l^q)$ is a prior or a base probability for describing the random endowment; the belief $\pi(\omega_l^q)$ is an apriori probability for explaining how the individual adjust his or her perception of demand under the realized price.

One may raise a concern about the practical meaning of the specification (10). It is true that one may hardly set up an exact probabilistic form as that of (10) in practice. When one knows little about the law of the endowment $\mathbb{P}(\omega_l^q)$ of oneself, one of the general beliefs is to set the probability of his or her demand power to $1/n_l$. Later, Corollary 1 shows that the specification of (10) can approximate this type of the naive belief under some optimal criterion. Thus, even though a naive belief is not exactly the belief in (10), the current specification is an optimal representative candidate for the naive one.

5.2 Collective beliefs in the sector

The expected power of the total demand in the l-th sector is

$$\begin{aligned}\int e^{-p_l X^d(\omega_l)} \mathrm{d}\mathbb{P}(\omega_l) &= \int \cdots \int \exp\left(-p_l \sum_{q=1}^{n_l} X^d(\omega_l^q)\right) \mathrm{d}\mathbb{P}(\omega_l^1) \cdots \mathrm{d}\mathbb{P}(\omega_l^{n_l}) \\ &\overset{(*)}{=} \mathbb{E}\left[e^{-p_l X^d(\omega_l^1)}\right] \cdots \mathbb{E}\left[e^{-p_l X^d(\omega_l^{n_l})}\right] = e^{-\sum_{q=1}^{n_l} \psi_q(p_l)},\end{aligned}$$

where $\mathbb{P}(\omega_l)$ as shown in (2) is a convolution of $\mathbb{P}(\omega_l^1)$, ..., $\mathbb{P}(\omega_l^{n_l})$. The equality $\overset{(*)}{=}$ holds because the individual demands $\{X^d(\omega_l^q)\}_{q=1,\dots,n_l}$ are independent.

Similarly, one can deduce the aggregation of individual beliefs in the sector level

$$\pi(\omega_l) = \exp\left\{-p_l X_l^d + \psi(p_l)\right\} \tag{11}$$

where $\psi(p_l) = \sum_{q=1}^{n_l} \psi_q(p_l)$. The probability functions $\pi(\omega_l^q)$ and $\pi(\omega_l)$, like $\mathbb{P}(\omega_l^q)$ and $\mathbb{P}(\omega_l)$, have the infinite divisible property. Also, the functions of $\pi(\omega_l^q)$ and $\pi(\omega_l)$ demonstrate that they are also in the (canonical) exponential family.[4]

One important implication of $\pi(\omega_l)$ is the invariant of expected sector demand x_l^d:

$$\begin{aligned}\frac{\partial \psi(p_l)}{\partial p_l} &= -\frac{\partial}{\partial p_l}\left\{\log \mathbb{E}\left[e^{-p_l X_l^d}\right]\right\} \\ &= \frac{\int X^d(\omega_l) e^{-p_l X^d(\omega_l)} \mathrm{d}\mathbb{P}(\omega_l)}{e^{-\psi(p_l)}} = \int X^d(\omega_l)\pi(\omega_l)\mathrm{d}\mathbb{P}(\omega_l) \\ &= \int X^d(\omega_l)\mathrm{d}\mathbb{Q}(\omega_l) = x_l^d,\end{aligned}$$

[4]The exponential family is one of most crucial parametric distribution family. It has been widely used in statistics, machine learning, artificial intelligence, etc as a baseline model. These conceptual similarities may hint about a more profound overlap to be explored.

where $\mathrm{d}\mathbb{Q}(\omega_l) = \pi(\omega_l)\mathrm{d}\mathbb{P}(\omega_l)$ is an adaptive probabilistic law. Because the expected sector demand x_l^d is utilized in the pricing system $p_l(\mathbf{x}) \in \langle(\mathbf{f}, \mathbf{x}^d)\rangle_{\mathbb{C}}$ where $\mathbf{x}^d$ includes x_l^d, if x_l^d varies, the price of this element would also change. For maintaining the same price level p_l, the new expected sector demand $\int X^d(\omega_l)\mathrm{d}\mathbb{Q}(\omega_l)$ should stay at the original demand level. Thus $\pi(\omega_l)$ is attached to an additional constraint of its expected value, while $\pi(\omega_l^q)$ has no such a constraint. Both $\pi(\omega_l^q)$ and $\pi(\omega_l)$ are still in the same distribution family. This difference arises due to the different roles of individual demands and sector demands. The sector demands would affect the price level of this sector, while an individual demand can only create a significant impact on the pricing system when it has a vital enough fraction of the sector demand.

The exponent of the sector demand power depends on price since the derivative of this exponent gives the expected sector demand. A natural question is what association does the expected sector demand make with the price. The following theorem gives an informatics explanation of $\pi(\omega_l)$. In this explanation, the price to one sector means the tendency of sticking to the expected sector demand. On the demand side, the price is viewed as a shadow price to the fixed expected demand constraint.

Theorem 3. *(Shadow price) The collective belief $\pi(\omega_l)$ given in (11) maximizes the the average amount of information, or maximize the entropy relative to $\mathbb{P}(\omega_l)$, in the l-th sector*

$$\begin{aligned} &\max_{\mathbb{Q}} \int \left[\log \frac{\mathrm{d}\mathbb{Q}(\omega_l)}{\mathrm{d}\mathbb{P}(\omega_l)}\right] \mathrm{d}\mathbb{Q}(\omega_l) \\ &s.t. \quad \int X^d(\omega_l)\mathrm{d}\mathbb{Q}(\omega_l) = x_l^d, \ \int \mathrm{d}\mathbb{Q}(\omega_l) = 1 \end{aligned} \tag{12}$$

if the Lagrangian multiplier, or the shadow price, of the constraint on the expected demand x_l^d is the same as the price p_l in this sector.

The meaning of the price given in Theorem 3 is subjective. The collective sector belief is composed by individual beliefs. Individuals cannot control the expected sector demand. But if the price is unchanged, individuals would expect the sector demand remains unchanged, and meanwhile, they would maximize the available information. The consequence of such behaviors makes the collective belief attempt to utilize the price, rather than the expected demand, to guide the evolution of the market power. Thus in the sector belief, the price represents the gained entropy by an infinitesimal unit of change of the expected demand in this sector. Because entropy is an indicator of information complexity, the maximum entropy implies the willingness of acquiring as much information as possible. The more complex the market is, the more opportunity one can seize for optimizing one's power.

5.3 Remarks

The objective function in (12) can be interpreted as the Kullback–Leibler divergence between two probabilities. It, in fact, measures the dissimilarity between these probabilities. The following corollary shows that the exponential family probability in (10) is an optimal representation for the naive belief $1/n_l$ as their dissimilarity attains the minimum in the Kullback-Leibler divergence.

Corollary 1. *(Naive belief) In the l-th sector, let $\bar{\pi}_q^*$ be the closest belief to the naive individual belief $1/n_l$ in terms of Kullback-Leibler divergence. When n_l increases, the belief $\bar{\pi}_q^*$ converges to $\pi(\omega_q^l)$ in (10).*

Corollary 1 implies that the exponential family induced by the market power is an approximating form to the standard naive belief. Especially when the sector demand is of many individuals, the probabilistic belief given in (10) is almost the same as the naive belief. That is, an individual who holds the naive shares the similar perception towards the market as those who hold the beliefs based on calculating their market power. In this case, two types of beliefs may generate similar market actions. By this principle, any belief that is close to $1/n_l$ can be replaced by $\pi(\omega_q^l)$.

Another remark for the exponential family is that it may create an illusion that there is a universal belief. However, this seemingly universal form of beliefs is merely a device to represent the other beliefs that are close to it. This illusion looks harmless when we consider only the naive type of beliefs. However, $\pi(\omega_q^l)$ and $\pi(\omega_q)$ are beliefs of the partially observed powers. The cyclic patterns from the real-valued powers are neglected. Following the construction of the belief in Definition 5, an oracle belief of a "prophet", a hypothetical person who can recognize the imaginary price, should be

$$\frac{\cos(-y_l X^d(\omega_l^q))e^{-p_l X^d(\omega_l^q)}}{\mathbb{E}[\cos(-y_l X^d(\omega_l^q))e^{-p_l X^d(\omega_l^q)}]}. \tag{13}$$

When the imaginary price y_l is small, $\cos(\cdot)$ approximates to one so that the belief in (13) is very close to $\pi(\omega_q^l)$. In this case, the belief in (13) and $\pi(\omega_q^l)$ are indistinguishable. Then, individual with $\pi(\omega_q^l)$ may be confident with his or her belief because it is consistent with the underlying truth belief. When the value of the imaginary price is getting larger, an individual may discover the subtle differences, but since the imaginary price is not observable, this individual can do nothing with the belief. This phenomenon creates another illusion. In this illusion, one may notice the imperfection of one's belief, but one is incapable of building up a new belief, meanwhile one would compromise with the imperfect belief because occasionally it coincides with the perfect one.

Conceptually, an individual belief only forms the individual's action. It doesn't affect the market price. The sector belief is a collection of individual beliefs with a fixed expected sector demand. In principle, it doesn't affect the market price either. However, this may not be the case. An individual may stick to the illusion of his or her belief, but once the imperfections propagate and the illusions collapse, the collective belief in the sector may realize the mismatch. Then a force would be formed to move to a new market status, for example, a new expected sector demand. Though the force of supply and demand, such a movement may eventually disturb the price from its equilibrium position. As the price is a function interacting across K sectors, we consider this issue when the collective belief that is extended to the market level.

6 Collective beliefs in the market

When the focus is shifted to the market level, the formation of a belief attaches to more conditions. These conditions are mainly about the equilibrium. Recall that the equilibria $\mathbf{x} \in \mathbb{V}(\mathbf{f}, \mathbf{x}^d)$ belong to a variety in a pure exchange economy and the prices $\mathbf{h} \in \mathbb{I}_{\mathbb{C}}(\mathbb{V})$ belong to an ideal of these varieties. When the sector demands $\mathbf{x}^d$ vary, the variation will first influence the supplies $\mathbf{x} \in \mathbb{V}(\mathbf{f}, \mathbf{x}^d)$ then the prices $\mathbf{h} \in \mathbb{I}_{\mathbb{C}}(\mathbb{V})$. The new realized prices $\mathbf{p}$ from $\mathbf{h}$ deliver new visions of the market belief. Thus, unlike the individual beliefs and the sector beliefs, the collective market beliefs need to take the equilibrium conditions into account. Another feature on the market level is that if the complex valued price vector $\mathbf{h} \in \mathbb{I}_{\mathbb{C}}(\mathbb{V})$ can be recognized, it is possible to form an oracle belief that is consistent with the underlying truth. But even in the free market, the oracle belief may not preserve the equilibrium. Such a deficiency may be due to an imaginary price or a singularity.

6.1 Belief and the equilibrium free market

Definition 6. (Belief and oracle belief of the free market) Let the economic element $\mathbf{x} \in \mathbb{V}(\mathbf{f}, \mathbf{x}^d)$ and the price $\mathbf{h} \in \mathbb{I}_{\mathbb{C}}(\mathbb{V})$ be the equilibrium values. The market equilibrium induces a unit expected market power

$$\int e^{\mathbf{h}^\top(\mathbf{f}-\mathbf{X}^d)} \mathrm{d}\mathbb{P}(\omega_1, \dots, \omega_K) = \int \Pi(\Omega) \mathrm{d}\mathbb{P}(\Omega) = 1$$

where $\mathbb{P}(\omega_1, \dots, \omega_K) = \mathbb{P}(\Omega)$ is a joint distribution of K sectors' total endowments associating with $\mathbf{X}^d = (X^d(\omega_1), \dots, X^d(\omega_K))$. The oracle belief is $\Pi(\Omega)$. When only the real-valued prices $\mathbf{p}$ are observed, the collective market belief is $\Pi(\Omega) = e^{\mathbf{p}^\top(\mathbf{f}-\mathbf{X}^d)}$.

The expected market power of the equilibrium status is a unit $\mathbb{E}[e^{\mathbf{h}^\top(\mathbf{f}-\mathbf{X}^d)}] = e^0$ because the market clear condition under the invisible hand implies a unit, an invariant base value. The uncertainty of the endowments drives the participants to establish a unit value for the baseline expectation. The unit e^0 is this baseline. The market oracle belief $\Pi(\Omega)$ depends on the equilibrium condition. The unit, one can deduce $\mathbb{E}[e^{-\mathbf{h}^\top\mathbf{X}^d}] = e^{-\mathbf{h}^\top\mathbf{f}}$ as the expectation is only taken with respect to $\mathbb{P}(\Omega)$ for $\mathbf{X}^d$. Following the same construction as those of individual beliefs, we have another expression of $\Pi(\Omega)$ in Definition 6

$$\frac{\exp\left(-\mathbf{h}^\top\mathbf{X}^d\right)}{\mathbb{E}\left[\exp\left(-\mathbf{h}^\top\mathbf{X}^d\right)\right]} = \exp\left(-\mathbf{h}^\top\mathbf{X}^d + \mathbf{h}^\top\mathbf{f}\right) = \Pi(\Omega).$$

The probabilistic law $\mathbb{P}(\Omega)$ across K-sectors is a joint probability as all the sectors can interact. The infinite divisible property relies on the independent identical components. Thus the infinite divisibilities of $\mathbb{P}(\omega_l^q)$ and $\mathbb{P}(\omega_l)$ are not automatically extended to $\mathbb{P}(\Omega)$. The structure follows

$$\overbrace{\overbrace{\mathbb{P}(\omega_1^1)\star\cdots\star\mathbb{P}(\omega_1^{n_1})}^{\mathbb{P}(\omega_1)}, \dots, \overbrace{\mathbb{P}(\omega_K^1)\star\cdots\star\mathbb{P}(\omega_K^{n_K})}^{\mathbb{P}(\omega_K)}}^{\text{equilibrium } \mathbb{P}(\Omega)}.$$

Without the infinite divisibility, the expected market power $\mathbb{E}[e^{-\mathbf{h}^\top\mathbf{X}^d}]$ may have an unattainable form. The condition of the unit expected market power in Definition 6 gives the feasible expression of $\Pi(\Omega)$.

In particular, when the imaginary prices vanish $\mathbf{y} = 0$, there is a one-to-one logarithm transform exists for Π so that the equilibrium condition or the market clear condition is achieved

$$\int \left[\log \Pi(\Omega)\right] \mathrm{d}\mathbb{P}(\Omega) = \mathbb{E}\left[-\mathbf{p}^\top\mathbf{X}^d\right] + \mathbf{p}^\top\mathbf{f} = -\mathbf{p}^\top(\mathbf{x}^d - \mathbf{f}) = 0.$$

The logarithm transform make the invisible hand dysfunctional so that the market belief can map inversely back to the original realistic values. When $\mathbf{y} = 0$, this belief can also induce a series of market demands that eventually converges to the equilibrium market demand.

Theorem 4. *(Global optimization) When the production* $\mathbf{f}(\mathbf{x})$ *is fixed, the imaginary prices are zero and the equilibrium equations satisfying some regularity conditions*[5]*, the market belief is a global optimization law of clearing the market. There exists a vector of stochastic processes* $\mathbf{X}^d(t)$ *initiated by this market belief and it executes the optimization such that* $\mathbf{X}^d(t) \to \mathbf{x}^d$ *when* $t \to \infty$.

The intuition behind Theorem 4 is based on the random search or the simulated annealing algorithm. A stochastic law is used for guiding the searching process. In this case, the stochastic law depends on the market belief. The searching process evaluates different states of values $\mathbf{X}^d(t)$ across the time t. Eventually, $\mathbf{X}^d(t)$ will converge to the state or states that can achieve the highest probability. By the unit market power in Definition 6, when $\mathbf{X}^d = \mathbf{x}^d$, the probability is a constant unit. Thus, the searching process will converge to $\mathbf{x}^d$. Theorem 4 provides a global approach to explain how a series of demand shocks can clear the market when the imaginary price is non-existent.[6]

When $\mathbf{y} \neq 0$, the logarithm transform for the complex valued $\Pi(\Omega)$ can be defined as follows

$$\log \Pi(\Omega) = -\mathbf{p}^\top(\mathbf{X}^d - \mathbf{f}) - \mathrm{i}\left[\mathbf{y}^\top(\mathbf{X}^d - \mathbf{f}) \bmod 2\pi\right].$$

The modular operator restricts the imaginary part to a fixed interval with a length of 2π.[7] The logarithm and exponential function are no longer in a one-to-one transformation, and in general $\log \Pi(\Omega)$ is not the inversion of $e^{\mathbf{h}^\top(\mathbf{f}-\mathbf{X}^d)}$. Then expectation of $\log \Pi(\Omega)$ does not equal to zero due to the imaginary term, and hence in principle, the economy is not in the equilibrium. But the real-valued term of $\mathbb{E}[\log \Pi(\Omega)]$ is zero which means that the expected inversion of the belief intends to clear the market. Thus if the imaginary prices do not vanish, even the oracle belief of the market cannot generate a law by which the economy converges to the equilibrium.

Here is a remark for the contradiction roles of an oracle belief. The market equilibrium demand $\mathbf{x}^d$ is an underlying true equilibrium state that holds for both real-valued and the

[5] The conditions are listed in the Appendix A.7.

[6] Another way to look at the problem is to consider it as a toric model in the algebraic geometry. The exponential family is a log-linear model. The likelihood functions of this model are called the toric models.

[7] From the Euler's formula, it is known that any $z \in \mathbb{C}$ can be represented as $z = |z|e^{\mathrm{i}\theta}$ where $-\pi < \theta < \pi$. The logarithm is $\log|z| + \mathrm{i}\theta$.

imaginary-valued parts, while the belief $\Pi(\Omega)$ can only treat $\mathbf{x}^d = \mathbf{f}$ as one of the possible states making real-valued part zero. If the oracle belief $\Pi(\Omega)$ generates the law of the economy, then $\mathbf{x}^d$ is not a steady equilibrium point for this law. The result of global optimization as in Theorem 4 is not necessarily true even if an oracle belief observes the imaginary price.

Theorem 4 gives limited support to the free market doctrine. With the invisible hands, the belief $\Pi(\Omega)$ of the free market indeed can enhance the market to reach its equilibrium status, but this goal is achieved only if the imaginary prices play no role in determining the equilibrium states. In general, the free market belief, even it perfectly recognizes the imaginary prices, cannot place a market economy into its equilibrium states. Thus the traditional free-market doctrine crushes when the imaginary dimension enters the economy. This finding also coincides with the reality that seldom if ever empirical evidence supports an equilibrium state arrived in a free market-oriented economy.

6.2 Belief and the singularity

The previous subsection discusses one possibility of the deviation from the equilibrium. That is caused by the ill-defined law generated by a belief involving complex values. When facing an uncertain growing world, the individuals and their coalitions amplify their reactions, and after that, the beliefs may induce a more significant distortion of the proper movements which is known as a singularity. The following subsection will study another possibility of disequilibrium, the emergence of singularity and will discuss whether these moves would bring a catastrophe or an establishment to the economy.

Theorem 5. *(Extendable market) The economy defined by Definition 1 is extendable. Suppose that the set of current equilibria $\mathbb{V}(\mathbf{f}_1, \mathbf{x}_1^d) \subset \mathbb{R}^k$ with economic elements $x_1, \dots x_k$, and the equilibria in another exchange economy are in the set $\mathbb{V}(\mathbf{f}_2, \mathbf{x}_2^d) \subset \mathbb{R}^{K-k}$ with new elements $x_{k+1}, \dots, x_K$. Then the equilibria of combining two economies are in the Cartesian product*

$$\mathbb{V}(\mathbf{f}_1, \mathbf{x}_1^d) \times \mathbb{V}(\mathbf{f}_2, \mathbf{x}_2^d) \subset \mathbb{R}^k \times \mathbb{R}^{K-k} = \mathbb{R}^K$$

which is $\mathbb{V}(\mathbf{f}_1, \mathbf{f}_2, \mathbf{x}^d)$. The pricing function $\mathbf{h}(\mathbf{x})$ for $\mathbb{V}(\mathbf{f}_1, \mathbf{f}_2, \mathbf{x}^d)$ is defined on $\mathbb{C}^K[\mathbf{x}]$ while $\mathbf{h}_1(\mathbf{x}_1)$ for $\mathbb{V}(\mathbf{f}_1, \mathbf{x}_1^d)$ is defined on $\mathbb{C}^k[\mathbf{x}_1]$ and $\mathbf{h}_2(\mathbf{x}_2)$ for $\mathbb{V}(\mathbf{f}_2, \mathbf{x}_2^d)$ is defined on $\mathbb{C}^{K-k}[\mathbf{x}_2]$.

The theorem implies that one small market $(\mathbb{V}(\mathbf{f}_1, \mathbf{x}_1^d), \mathbf{h}_1)$ can be extended to a bigger one $(\mathbb{V}(\mathbf{f}_1, \mathbf{f}_2, \mathbf{x}^d), \mathbf{h})$ by absorbing some economic elements from the other market $(\mathbb{V}(\mathbf{f}_2, \mathbf{x}_2^d), \mathbf{h}_2)$. Thus new elements can integrate with the existing ones. The extension process for the elements is simply an integration of two sets. But for the pricing function, the new market needs to re-evaluate the prices of all elements simultaneously. The re-evaluation of the prices induces a new belief in the market.

Let $\Pi(\Omega_1)$ and $\Pi(\Omega_2)$ be the beliefs of the original markets $(\mathbb{V}(\mathbf{f}_1, \mathbf{x}_1^d), \mathbf{h}_1)$ and $(\mathbb{V}(\mathbf{f}_2, \mathbf{x}_2^d), \mathbf{h}_2)$ and let $\Pi(\Omega)$ be the new market beliefs for $(\mathbb{V}(\mathbf{f}_1, \mathbf{f}_2, \mathbf{x}^d), \mathbf{h})$. The extension process for the beliefs consists of two steps, the integration step, and the coupling step. First, it merges two old beliefs $\Pi(\Omega_1)$ and $\Pi(\Omega_2)$. This leads to a mixture probability $\Pi_0(\Omega)$,

$$\Pi_0(\Omega) = (1-\gamma)\Pi(\Omega_1) + \gamma\Pi(\Omega_2) = (1-\gamma)\frac{e^{-\mathbf{h}_1^\top \mathbf{X}_1^d}}{e^{-\mathbf{h}_1^\top \mathbf{f}_1}} + \gamma\frac{e^{-\mathbf{h}_2^\top \mathbf{X}_2^d}}{e^{-\mathbf{h}_2^\top \mathbf{f}_2}}$$

where $0 \le \gamma < 1$. Second, it couples this mixture belief with the one suited for the new economy

$$\mathbb{T}\left[\Pi_0(\Omega)\right] = \exp\left(-\mathbf{h}^\top \begin{bmatrix} \mathbf{X}_1^d \\ \mathbf{X}_2^d \end{bmatrix}\right) / \exp\left(-\mathbf{h}^\top \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \end{bmatrix}\right) = \Pi(\Omega) \tag{14}$$

where $\mathbb{T}[\cdot]$ means a transportation operator from the mixture probability $\Pi_0(\Omega)$ to the new belief $\Pi(\Omega)$.[8] The extension of the market belief now is reduced to a coupling technique of the mixture belief $\Pi_0(\Omega)$ to the new belief $\Pi(\Omega)$.

The singularity often occurs in the coupling procedure of these beliefs. To avoid technical details, here I only discuss the intuition of the singularity problem. The transportation operator requires a smoothness property for its gradient or an approximate gradient.[9] Unfortunately, the mixture probability is known to have some problems of the collection of its partial derivatives. The linear dependence gives the non-smoothness of the derivatives even if the imaginary prices vanish $\mathbf{y} = 0$. If the partial derivatives are not linearly independent, that is

$$c_1 \partial\Pi(\Omega_1) + c_2 \partial\Pi(\Omega_2) = 0$$

for some $c_1, c_2 \neq 0$, then the singularity appears in the Jacobian matrix and the Hessian matrix of $\Pi_0(\Omega)$.

When the coupling procedure becomes non-smooth, the extended market belief is not equivalent to any of $\Pi(\Omega_1)$, $\Pi(\Omega_2)$, or $\Pi(\Omega)$ because of the failure of the coupling. Then the

[8]The model associated with the transportation problem (14) is a minimization of the distance between $\Pi_0(\Omega)$ and $\Pi(\Omega)$: $\min_{\mathbf{h},\gamma} \mathrm{d}\left(\Pi_0(\Omega), \Pi(\Omega)\right)$ for some distance function $\mathrm{d}(\cdot,\cdot)$. One typical choice for this distance function is the Wasserstein distance. Then the transportation problem (14) becomes the optimal transportation problem.

[9]For example, one standard choice for $\mathbb{T}[\cdot]$ is $\mathbb{T}[\cdot] = \exp(\tilde{\nabla}(\cdot))$, where $\tilde{\nabla}(\cdot)$ is the approximate gradient of some function. Please see [11, Theorem 10.41] for details.

pricing mechanism also becomes out of order since neither the old prices $\mathbf{h}_1$, $\mathbf{h}_2$ nor the new prices $\mathbf{h}$ can make the market equilibrium. This issue can be enlarged to the situation of market failure.

The linearly dependent partial derivatives of the beliefs can also happen when some of the new elements are believed to replace the old ones by the market. The singularity could be a natural case for some innovative elements or for some elements that are unfamiliar to the market participants. In this case, the mixture belief is hardly guided by the market prices. In contrast, the mixture market belief causes the distorted amount of demands which twists the pricing signals, and after that, the following pricing processes are all misguided by the beliefs. Eventually, if the new elements $\mathbf{X}_2^d$ indeed replace the old ones $\mathbf{X}_1^d$, the linear dependence will disappear as $\mathbf{X}_2^d$ emerges. This singularity becomes an innovation process. On the other hand, if the new elements $\mathbf{X}_2^d$ turn out to be a duplication of $\mathbf{X}_1^d$, the linear dependence will also disappear as $\mathbf{X}_2^d$ are redundant. Then this singularity only induces a pricing bubble to the market.

For rigorous discussions about the transportation problem, please refer to [11] and [5]. The statistical conditions about singularities in the exponential family are discussed in [8]. Resolution of singularities is rather technical and is beyond the current scope. However, I would like to point out that the standard resolution method of introducing new variables to a singular system (blowing-up) is feasible in the current context. Although Theorem 5 is given for two separate markets, the result can be applied a recursive integration of any amount of markets. Thus globalization, as a natural way of expanding the market, may be thought of as a resolution of singularity. However, this argument only holds when the new elements do not introduce new singularities.

7 Market Growth

7.1 Adaptive belief about the growth

In the previous sections, it has been shown that the market belief can only induce the market equilibrium when the imaginary prices are non-existent. Also, individual beliefs and sector beliefs do not corporate with the oracle market belief. So these beliefs do not exactly stay on the path of being equilibrium. In the real world, not matter whether the underlying states are in the equilibrium stage, there are beliefs formed by the individuals, the sectors, and the markets.

This sub-section considers a growing market and uses the individual beliefs developed before to demonstrate their roles in the market growth. The probabilistic beliefs are more flexible than the polynomial structural characterizations of equilibria. With their beliefs, the participants can follow the trends of the growth without bothering to know the underlying equilibria. Participants adapt their beliefs to the new prices and update their visions of the market. As long as the reasoning of the adaption is according with the conscious, the beliefs can cope with dis-equilibrium situations. This provides an alternative view of how the beliefs work beyond the equilibrium concern.

The adaptive belief is given by conditioning of different arguments in the probability function. Note that the price p_l of $\pi(\omega_l^q)$ defined in (10) is treated as a conditioning value. One can reformulate the individual belief $\pi(\omega_l^q)$ of demand as a belief of price by interchanging the roles of the variable and the conditioning value. The belief of price is

$$\pi(p_l|\omega_l^q) = \frac{e^{-p_l X^d(\omega_l^q)}\pi(p_l)}{\int e^{-p_l X^d(\omega_l^q)}\pi(p_l)\mathrm{d}p_l} = \frac{\pi(\omega_l^q)\pi(p_l)}{\int \pi(\omega_l^q)\pi(p_l)\mathrm{d}p_l} \tag{15}$$

where $\pi(p_l)$ is an arbitrary prior belief of the price. The expression of (15) is the Bayes' law. One can use a simplified expression $\pi(p_l|\omega_l^q) \propto \pi(\omega_l^q)\pi(p_l)$ to emphasize two beliefs are formed proportionally.

The belief of price is a subjective item. In the market, an individual will observe the real-valued price. Then the belief of demand will be updated by conditioning the new price value. Similarly, the adaptive belief of price will be updated by the Bayes' law, and the previous belief of price becomes the prior belief in (15). The last step implies $\pi(p_l) = \pi(p_l|\omega_l^q)$. This sequential procedure can be summarized as follows

Initiation: The prior belief $\pi(p_l)$ is established.

Demand-step: By observing price p_l, the belief of demand $\pi(\omega_l^q)$ is established.

Price-step: By observing demand $X^d(\omega_l^q)$, the belief of price $\pi(p_l|\omega_l^q) \propto \pi(\omega_l^q)\pi(p_l)$ is established.

Loops: Set $\pi(p_l) = \pi(p_l|\omega_l^q)$, go to the Demand-step.

It is pointed out in [6, Theorem 5] that a Gamma distribution can capture a general growth pattern. As the Gamma distribution belongs to the exponential family and satisfies the infinite divisible property, I use it to illustrate this belief adaption procedure.

Let the individual belief $\pi(\omega_l^q)$ follow a Gamma distribution such that[10]

$$\pi(\omega_l^q) = \frac{(p_l X^d(\omega_l^q))^{\alpha_q} e^{-p_l X^d(\omega_l^q)}}{X^d(\omega_l^q)\Gamma(\alpha_q)}$$

where α_q and p_l are the parameters of this distribution, $\Gamma(\cdot)$ is the Gamma function. The parameter α_q is to describe the expected demand level for this individual. The distribution can be written as Gamma(α_q, p_l). The sector belief is Gamma$(\bar{\alpha}_l, p_l)$ where $\bar{\alpha}_l = \sum_{q=1}^{n_l} \alpha_q$ follows the infinite divisibility of Gamma distribution. It can be expressed as

$$\pi(\omega_l) \propto (p_l)^{\bar{\alpha}_l} (X_l^d)^{\bar{\alpha}_l - 1} \exp\left\{-p_l X_l^d\right\}.$$

Thus the meaning of $\bar{\alpha}_l$ is the expected demand in the l-th sector, $\bar{\alpha}_l = x_l$. It is known that $\mathbb{E}[\pi(\omega_l)] = x_l/p_l$ by the property of Gamma distribution. It implies that the expected sector belief is a ratio between the expected demand and the price of this sector. Either increasing the price or decreasing the expected demand will decrease the value of the expected belief. This rule fits the equilibrium analysis of the demand and supply relation.

The prior belief of price should be proportional to $\pi(\omega_q)$. Thus it is assumed to follow a Gamma(α_0, β_0) distribution such that

$$\pi(p_l) \propto \beta_0^{\alpha_0} (p_l)^{\alpha_0 - 1} \exp\left\{-\beta_0 p_l\right\}$$

where α_0 represents previous expected demand of the sector and β_0 represented previous realized demand of the sector. The belief of price $\pi(p_l|\omega_l)$ is the product of $\pi(\omega_l)$ and $\pi(p_l)$. It is proportional to

$$\pi(p_l|\omega_l) \propto (p_l)^{\bar{\alpha}_l + \alpha_0 - 1} \exp\left\{-p_l(X_l^d + \beta_0)\right\}$$

which is a Gamma$((\alpha_0 + \bar{\alpha}_l), \beta_0 + X_l^d)$ distribution. Thus the belief of price is characterized by the updated expected demand and the realized demand of the l-th sector.

[10] It can be presented in terms of the canonical exponential family form

$$\frac{e^{-p_l X^d(\omega_l^q)}}{X^d(\omega_l^q)\Gamma(\alpha_q) e^{-\alpha_q \log(p_l X^d(\omega_l^q))}} = \frac{s(X^d(\omega_l^q))(e^{-p_l X^d(\omega_l^q)})}{\mathbb{E}[s(X^d(\omega_l^q)) e^{-p_l X^d(\omega_l^q)}]}$$

where $s(X^d(\omega_l^q)) = X^d(\omega_l^q) e^{-\alpha_q \log(p_l X^d(\omega_l^q))}$ is the normalizer and $\mathbb{E}[s(X^d(\omega_l^q)) e^{-p_l X^d(\omega_l^q)}] = \Gamma(\alpha_q)$.

7.2 Belief about the market structure

It is shown in [6, Section 6] that the growth process with the Gamma distribution can be divided into two types: α-growth and β-growth. Each type of the growth depends on the increase of the corresponding parameter. In the previous example, the belief of price has both the α-growth and the β-growth because the parameter α_0 increases to $\alpha_0 + \bar{\alpha}_l$ and the parameter β_0 increases to $\beta_0 + X_l^d$. The α-growth leads to a more equal situation than the β-growth, so the α-growth is of greater interest. The example shows that the price only affects the belief of sector demand $\pi(\omega_l) \sim \mathrm{Gamma}(\bar{\alpha}_l, p_l)$ through the β-growth channel, while the α-growth of the belief is contributed by the expected sector demand.

A new belief about the growth of the market structure will be proposed by the result of the Gamma growth. Consider only the α-growth impact to this belief. The influence of the prices across the sectors needs to be tackled. Consider the α-growth regarding the profit $W_i = X_i p_i$ in each sector i. As the belief of X_i follows $\mathrm{Gamma}(\bar{\alpha}_i, p_i)$, by the property of Gamma distribution, the belief of the profit $W_i = X_i p_i$ follows the distribution $\mathrm{Gamma}(\bar{\alpha}_i, 1)$.

Theorem 6. *(Market structural belief) Let the beliefs of profit $W_1, \ldots W_K$ across K-sectors follow $\mathrm{Gamma}(\bar{\alpha}_i, 1)$ for $i = 1, \ldots, K$. The market share of each i sector is given by*

$$M_i = \frac{W_i}{\sum_{j=1}^K W_j}, \quad i = 1, \ldots, K.$$

Then (i) the belief of $(M_1, \ldots, M_K)$ is a Dirichlet distribution $\mathrm{Dir}(\bar{\alpha}_1, \ldots \bar{\alpha}_K)$ and the belief of $(M_1, \ldots, M_K)$ is independent of the total profit $\sum_{i=1}^K W_i$ in the market. (ii) in particular, let $M_1 = 1 - \sum_{i=2}^K M_K$, the belief induces a system of stochastic differential equations for the vector $(M_1, \ldots, M_K)$ such that

$$\mathrm{d}M_i(t) = \frac{1}{2}(\bar{\alpha}_i M_1(t) - \bar{\alpha}_1 M_i(t))\mathrm{d}t + \sqrt{M_i(t)M_1(t)}\mathrm{d}B_i(t) \tag{16}$$

for $i = 2, \ldots K$ where $B_i(t)$ is a Brownian motion. The stationary distribution of (16) is $\mathrm{Dir}(\bar{\alpha}_1, \ldots \bar{\alpha}_K)$.

Theorem 6 (i) provides a closed function form for the belief of market structure. It is known that Dirichlet distribution is one of most flexible and fundamental priors in the Bayesian analysis. The implication of Theorem 6 (i) is to consider this belief as a prior in the empirical study. The parameters of α-growth across K-sectors can be empirical observable. Then the belief of the market structure can adapt to the new information as the procedure discussed in the previous subsection. In a different setting, [1, Ch. 9] considers the market structure to follow a Dirichlet distribution and uses it as the cornerstone to analyze the macro-behavior of the financial market with heterogeneous agents.

Theorem 6 (ii) models a system of processes of these market shares. The probabilistic law of these processes coincides with the structural belief. A specific example is given for a three-sector movement such that

$$\begin{aligned}
\mathrm{d}M_3(t) &= \frac{1}{2}(\bar{\alpha}_3 M_1(t) - \bar{\alpha}_1 M_3(t))\mathrm{d}t + \sqrt{M_3(t)M_1(t)}\mathrm{d}B_3(t), \\
\mathrm{d}M_2(t) &= \frac{1}{2}(\bar{\alpha}_2 M_1(t) - \bar{\alpha}_1 M_2(t))\mathrm{d}t + \sqrt{M_2(t)M_1(t)}\mathrm{d}B_2(t), \\
M_1(t) &= 1 - M_2(t) - M_3(t).
\end{aligned}$$

The above model is similar to several models of dynamical games with three bodies, for example, a stochastic prey-predator model or an SIR (Susceptible, Infectious, Recovered) model or an arms race model. As these models have been used to model competitions among species or interactions amongst patients or rivalries amongst countries, it is not surprising that the model in (16) is possible to visualize the variation of market shares across sectors. Thus the belief in Theorem 6 is not far from the existing strategies about interactions or competitions.

The parameters of the α-growth reflect the concentration of the belief over different sectors in this market. This concentration may be diluted when the market expands. Theorem 5 gives the possibility to extend the number of sectors as Dirichlet distribution. In principle, K can continuously increase, and the Dirichlet distribution can cope with this extension. However, when the market extends, new elements attract more attention and the demands in the old sectors may shrink. One utopian vision of a competitive market is that firms in all sectors have similar expected demands and the inequalities of the market structure across the sectors may disappear. The following corollary shows that when total expect demand across all sector is fixed, the belief of the market structure will not converge to a uniform distribution. But interestingly, the limit of this belief coincides with a distribution related to prime numbers. Conjecturally, a profile of sectors in a utopian market may correspond to a sequence of prime number factors.

Corollary 2. *The belief of $(M_1, \dots M_K)$ will not induce a uniformly distributed market even if $\bar{\alpha}_1 = \cdots = \bar{\alpha}_K = \frac{1}{K}$. In particular, as $K \to \infty$, this belief of the decreasing order of market* *shares converges to the distribution of*

$$\left\{ \frac{c_1 \log q_1}{\log n}, \ldots, \frac{c_K \log q_K}{\log n} \right\}$$

where n *is a positive integer satisfying* $n = q_1^{c_1} q_2^{c_2} ... q_K^{c_K}$, *a factorization of* n *into its constituent primes* $q_1, \ldots, q_K$.

8 Conclusive Remarks

When the mankind creates a realized substance, an imaginary one may coexist. The imaginary price is proved to be an invisible coexistence of the real-valued market price. The imaginary price, as the root of other imaginary forces in the market, establishes a superstructure medium. The medium is presented as various beliefs. Through these beliefs, the market empowers the participants, meanwhile the participants feel their powers, perceive how these powers affect the market movements, and act consistently with what they believe in. Under the influences of these imaginary forces, the participants execute their missions for the individual, the sector or the market.

The imaginary forces can also compile the gap between the non-equilibrium reality and the equilibrium illusion. Phenomena such as inconsistent expectations of the agents, innovations of the industries or cyclic patterns of the markets, that were thought to be attributed to the exogenous impacts, are now modeled in the unified framework that is consistent with the belief of equilibrium. The new alternative explanations challenge the free market doctrine and stimulate further analyses on the consequences of market structures regarding the imaginary forces. In sum, the added imaginary dimension paves a new path for approaching the reality in the market.

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A Proofs

A.1 The Proof of Theorem 1

Theorem. *(Hilbert's Nullstellensatz [2, Theorem 2 in Ch. 4]) Let* $\mathbb{F}$ *be an algebraically closed field. If* $g,\ f_1, \ldots, f_K \in \mathbb{F}[x_1, \ldots, x_K]$ *are such that* $g \in \mathbb{I}_{\mathbb{F}}(\mathbb{V})$*, then there exists an integer* $m \geq 1$ *such that*

$$g^m \in \langle f_1, \ldots, f_K \rangle_{\mathbb{F}}$$

and conversely is also true.

The proof of Theorem 1 makes use of the result from Hilbert's Nullstellensatz Theorem.

Proof. Given the field $\mathbb{R}$ of real numbers, the field of complex numbers $\mathbb{C}$ is the field extension of $\mathbb{R}$. The fundamental theorem of algebra says every non-constant $f \in \mathbb{C}[\mathbf{x}]$ has at least one complex root. Thus $\mathbb{C}[\mathbf{x}]$ is an algebraically closed field extension of $\mathbb{R}[\mathbf{x}]$.

If $\mathbb{V}(\mathbf{f}, \mathbf{x}^d)$ is empty, then $\mathbb{I}_{\mathbb{C}}(\mathbb{V}) = \mathbb{C}[\mathbf{x}]$ which is a vacuous truth in mathematical logic. So the empty $\mathbb{V}(\mathbf{f}, \mathbf{x}^d)$ is of no interest.

Note $f_1, \ldots, f_K \in \mathbb{R}[\mathbf{x}] \subset \mathbb{C}[\mathbf{x}]$ and $\mathbb{V}(\mathbf{f}, \mathbf{x}^d) \subset \mathbb{R}^K \subset \mathbb{C}^K$. Given the market clearing polynomial $g \in \mathbb{C}[\mathbf{x}]$ and a modified Definition 3, if $g \in \mathbb{I}_{\mathbb{C}}(\mathbb{V})$ then

$$\mathbb{I}_{\mathbb{C}}(\mathbb{V}) = \left\{ g \in \mathbb{C}[\mathbf{x}] : g(f_1(\mathbf{x}) - x_1^d, \ldots, f_K(\mathbf{x}) - x_K^d) = 0, \text{ for all } \mathbf{x} \in \mathbb{V}(\mathbf{f}, \mathbf{x}^d) \right\}.$$

By Hilbert's Nullstellensatz, there exists an integer $m \geq 1$ such that $g^m \in \left\langle f_1, \ldots, f_K, \mathbf{x}^d \right\rangle_{\mathbb{C}}$ for any $g \in \mathbb{I}_{\mathbb{C}}(\mathbb{V})$ where $\mathbf{x}^d$ here is a constant vector for the polynomials $\mathbf{f}$.

Since $\left\langle f_1, \ldots, f_K, \mathbf{x}^d \right\rangle_{\mathbb{C}}$ is set of price functions $\mathbf{h}$, by the modified Definition 2, there exists $\mathbf{h}$ such that

$$\sum_{l=1}^{K} h_l(\mathbf{x}) f_l(\mathbf{x}) - \sum_{l=1}^{K} h_l(\mathbf{x}) x_l^d = g^m(f_1(\mathbf{x}) - x_1^d, \ldots, f_K(\mathbf{x}) - x_K^d) = 0$$

for $h_1, \ldots, h_K \in \mathbb{C}[\mathbf{x}]$ and $h(\mathbf{x}) = \sum_{l=1}^{K} h_l(\mathbf{x})$. Thus for any $g \in \mathbb{I}_{\mathbb{C}}(\mathbb{V})$, we can find a corresponding $\mathbf{h} \in \left\langle f_1, \ldots, f_K, \mathbf{x}^d \right\rangle_{\mathbb{C}}$ and vice versa. The result follows. □

A.2 The Proof of Proposition 1

Proof. Note that for any $z \in \mathbb{R}$, the matrix $\mathbf{R} = \begin{pmatrix} \cos z & -\sin z \\ \sin z & \cos z \end{pmatrix}$ in (5) can be decomposed as

$$\begin{pmatrix} \cos z & -\sin z \\ \sin z & \cos z \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos z + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \sin y = \mathbf{1} \cos z + \mathbf{i} \sin z \tag{17}$$

where $\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\mathbf{i} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ are two basis matrices. It is easily checked that $\mathbf{1i} = \mathbf{i1} = \mathbf{i}$ and $\mathbf{i}^2 = -\mathbf{1}$. Thus $\mathbf{i}$ satisfies the axioms of being a complex number under addition and multiplication.

Next, we will show that

$$\begin{pmatrix} \cos z & -\sin z \\ \sin z & \cos z \end{pmatrix} = \exp \left\{ \begin{pmatrix} 0 & -z \\ z & 0 \end{pmatrix} \right\}. \tag{18}$$

For any $n \times n$ matrix $\mathbf{A}$, the exponential of $\mathbf{A}$ is given by

$$e^{\mathbf{A}} = \mathbf{1} + \mathbf{A} + \frac{\mathbf{A}^2}{2!} + \frac{\mathbf{A}^3}{3!} + \cdots .$$

Thus we have the expansion

$$\exp \begin{pmatrix} 0 & -z \\ z & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -z \\ z & 0 \end{pmatrix} - \begin{pmatrix} \frac{z^2}{2!} & 0 \\ 0 & \frac{z^2}{2!} \end{pmatrix} - \begin{pmatrix} 0 & \frac{-z^3}{3!} \\ \frac{z^3}{3!} & 0 \end{pmatrix} + \cdots .$$

The diagonal entities have the same form $\sum_{j=0}^{\infty} (-1)^j z^{2j}/(2j)!$ which is the Taylor series of $\cos z$. Similarly, one can show the off-diagonal terms are the Taylor series of $-\sin z$ and $\sin z$ respectively. Thus (18) is valid.[11]

[11] $\begin{pmatrix} 0 & -z \\ z & 0 \end{pmatrix}$ is called the skew-symmetric matrix, a simple Lie algebra $so(2)$. The exponential function $\exp \begin{pmatrix} 0 & -z \\ z & 0 \end{pmatrix}$ is a simple Lie group $SO(2)$. It is possible to extend the dual sentiment to any n types of sentiment. In such a case, a skew-symmetric matrix $\mathbf{S}$ needs to be an $n \times n$ real matrix with $\mathbf{S} + \mathbf{S}^T = 0$ and $\mathbf{SS}^T = \mathbf{I}$ where $\mathbf{I}$ is an $n \times n$ identity matrix.

As $\begin{pmatrix} 0 & -z \\ z & 0 \end{pmatrix} = \mathbf{i}z$, by (17) and (18), we have

$$\begin{pmatrix} \cos z & -\sin z \\ \sin z & \cos z \end{pmatrix} = \exp\left\{\begin{pmatrix} 0 & -z \\ z & 0 \end{pmatrix}\right\} = e^{\mathbf{i}z} = \mathbf{1}\cos z + \mathbf{i}\sin z.$$

By using the standard complex number notation, we have the result. □

A.3 Proof of Theorem 2

Proof. By Definition 4 and Proposition 1, there is

$$e^{-\mathbf{p}^\top \mathbf{X}} e^{-\mathrm{i}\mathbf{y}^\top \mathbf{X}} = e^{-\mathbf{p}^\top \mathbf{X}} \left(\cos(-\mathbf{y}^\top \mathbf{X}) + \mathrm{i}\sin(-\mathbf{y}^\top \mathbf{X})\right) = u(\mathbf{p}, \mathbf{y}) + \mathrm{i}v(\mathbf{p}, \mathbf{y}).$$

The partial derivatives of u and v w.r.t. $\mathbf{p}$ and $\mathbf{y}$ clearly exist and are continuous, so all that remains to show is that the Cauchy-Riemann equations are satisfied. We compute the partial derivatives of u and v:

$$\begin{aligned} \frac{\partial u}{\partial \mathbf{p}} &= -\mathbf{X}\left(e^{-\mathbf{p}^\top \mathbf{X}} \cos(-\mathbf{y}^\top \mathbf{X})\right), \frac{\partial v}{\partial \mathbf{y}} = -\mathbf{X}\left(e^{-\mathbf{p}^\top \mathbf{X}} \cos(-\mathbf{y}^\top \mathbf{X})\right), \\ \frac{\partial u}{\partial \mathbf{y}} &= \mathbf{X}\left(e^{-\mathbf{p}^\top \mathbf{X}} \sin(-\mathbf{y}^\top \mathbf{X})\right), \frac{\partial v}{\partial \mathbf{p}} = -\mathbf{X}\left(e^{-\mathbf{p}^\top \mathbf{X}} \sin(-\mathbf{y}^\top \mathbf{X})\right). \end{aligned}$$

Thus the above equations are satisfied since

$$\frac{\partial u}{\partial \mathbf{p}} = \frac{\partial v}{\partial \mathbf{y}}, \quad \frac{\partial u}{\partial \mathbf{y}} = -\frac{\partial v}{\partial \mathbf{p}}.$$

Taking the second derivatives of $\frac{\partial u}{\partial \mathbf{p}}$ and $\frac{\partial u}{\partial \mathbf{y}}$, we have

$$\begin{aligned} \frac{\partial^2 u}{\partial \mathbf{p} \partial \mathbf{p}^\top} &= \mathbf{X}^\top \mathbf{X} e^{-\mathbf{p}^\top \mathbf{X}} \cos(-\mathbf{y}^\top \mathbf{X}), \\ \frac{\partial^2 u}{\partial \mathbf{y} \partial \mathbf{y}^\top} &= -\mathbf{X}^\top \mathbf{X} e^{-\mathbf{p}^\top \mathbf{X}} \cos(-\mathbf{y}^\top \mathbf{X}). \end{aligned}$$

Similarly, taking the second derivatives of $\frac{\partial v}{\partial \mathbf{y}}$ and $\frac{\partial v}{\partial \mathbf{p}}$, we have

$$\frac{\partial^2 v}{\partial \mathbf{y} \partial \mathbf{y}^\top} = -\mathbf{X}^\top \mathbf{X} e^{-\mathbf{p}^\top \mathbf{X}} \sin(-\mathbf{y}^\top \mathbf{X}),$$

$$\frac{\partial^2 v}{\partial \mathbf{p} \partial \mathbf{p}^\top} = \mathbf{X}^\top \mathbf{X} e^{-\mathbf{p}^\top \mathbf{X}} \sin(-\mathbf{y}^\top \mathbf{X}).$$

Combining these second derivatives gives

$$\frac{\partial^2 u(\mathbf{p}, \mathbf{y})}{\partial \mathbf{p} \partial \mathbf{p}^\top} + \frac{\partial^2 u(\mathbf{p}, \mathbf{y})}{\partial \mathbf{y} \partial \mathbf{y}^\top} = 0, \; \frac{\partial^2 v(\mathbf{p}, \mathbf{y})}{\partial \mathbf{p} \partial \mathbf{p}^\top} + \frac{\partial^2 v(\mathbf{p}, \mathbf{y})}{\partial \mathbf{y} \partial \mathbf{y}^\top} = 0$$

which is a harmonic system. Also, according to the Cauchy-Riemann theorem, above equations are harmonic. □

A.4 Proposition 2

Proposition 2. *(Expected powers) Assume that (i) when* $p_l = 0$, $\mathbb{E}\left[e^{-p_l X^d(\omega_l^q)}\right] = 1$ *and (ii)* $\int_0^\infty \left[X^d(\omega_l^q)\right]^{-1} \mathrm{d}\mathbb{P}(\omega_l^q) < \infty$ *. The expected power of individual is*

$$\mathbb{E}\left[e^{-p_l X^d(\omega_l^q)}\right] = \int e^{-p_l X^d(\omega_l^q)} \mathrm{d}\mathbb{P}(\omega_l^q) = e^{-\psi_q(p_l)}$$

where $\psi_q(p_l) = \int_0^\infty \left[(1 - e^{-p_l X^d(\omega_l^q)})/X^d(\omega_l^q)\right] \mathrm{d}\mathbb{P}(\omega_l^q)$.

The proof uses the result of Laplace transform of infinitely divisible probability distribution.

Theorem. *[4, Theorem 1 in Ch.13]. The function* ω *is the Laplace transform of an infinitely divisible probability distribution if and only if* $\omega = e^{-\psi}$ *where* ψ *has (i) a completely monotone derivative and (ii)* $\psi(0) = 0$.

Proof. According to (1) and (2), $\mathbb{P}(\omega_l^q)$ belongs to the infinitely divisible probability family. The proof is to check $\psi_q(p_l)$ satisfies two conditions in above theorem. By the condition (ii) $\int_0^\infty 1/X^d(\omega_l^q) \mathrm{d}\mathbb{P}(\omega_l^q) < \infty$, the term

$$\psi_q(p_l) = \int_0^\infty \left[(1 - e^{-p_l X^d(\omega_l^q)})/X^d(\omega_l^q)\right] \mathrm{d}\mathbb{P}(\omega_l^q)$$

is bounded and continuous on $X^d(\omega_l^q)$. Consider the derivative of w.r.t. p_l

$$\frac{\mathrm{d}\psi_q(p_l)}{\mathrm{d}p_l} \overset{(a)}{=} \int_0^\infty \frac{\mathrm{d}\left[(1 - e^{-p_l X^d(\omega_l^q)})/X^d(\omega_l^q)\right]}{\mathrm{d}p_l} \mathrm{d}\mathbb{P}(\omega_l^q)$$

$$= \int_0^\infty e^{-p_l X^d(\omega_l^q)} \mathrm{d}\mathbb{P}(\omega_l^q).$$

Note that $\overset{(a)}{=}$ applies the Fubini theorem for continuous and bounded term within the big bracket. It is easy to see that $\mathrm{d}\psi_q(p_l)/\mathrm{d}p_l$ is completely monotone decreasing function on p_l. By the condition (i),

$$\mathbb{E}\left[e^{-p_l X^d(\omega_l^q)}\right] = 1 = e^0$$

when $p_l = 0$, so $\psi_q(0) = 0$. Thus $\psi_q(p_l)$ has a completely monotone derivative and $\psi_q(0) = 0$. □

A.5 Proof of Theorem 3

Proof. Note that $\mathrm{d}\mathbb{Q}(\omega_l) = \pi(\omega_l)\mathrm{d}\mathbb{P}(\omega_l)$. The Lagrangian of the problem

$$\begin{aligned}
&\max \int \left[\log \pi(\omega_l)\right] \pi(\omega_l)\mathrm{d}\mathbb{P}(\omega_l) \\
&\text{s.t.} \int X^d(\omega_l)\pi(\omega_l)\mathrm{d}\mathbb{P}(\omega_l) = x_l^d, \int \pi(\omega_l)\mathrm{d}\mathbb{P}(\omega_l) = 1.
\end{aligned}$$

is given as follows

$$J(\pi) = \int \left[\log \pi(\omega_l)\right] \pi(\omega_l)\mathrm{d}\mathbb{P}(\omega_l) - \lambda_0 \left(\int \pi(\omega_l)\mathrm{d}\mathbb{P}(\omega_l) - 1\right) + \lambda \left(\int X^d(\omega_l)\pi(\omega_l)\mathrm{d}\mathbb{P}(\omega_l) - x_l^d\right)$$

where the λ_0 and λ are the Lagrange multipliers. Note that $\int \left[\log \pi(\omega_l)\right] \pi(\omega_l)\mathrm{d}\mathbb{P}(\omega_l)$ is the entropy of $\pi(\omega_l)$.

The entropy attains an extremum when the functional derivative of $J(\pi)$ is equal to zero:

$$\begin{aligned}
\frac{\delta J}{\delta \pi \mathrm{d}\mathbb{P}(\omega_l)}(\pi) &= \ln \pi(\omega_l) + \frac{1}{\pi(\omega_l)}\pi(\omega_l) - \lambda_0 + \lambda X^d(\omega_l) \\
&= \ln \pi(\omega_l) + 1 - \lambda_0 + \lambda X_l^d = 0
\end{aligned}$$

This extremum is a maximum. Therefore, the maximum entropy probability distribution in this case must be of the form

$$\pi(\omega_l) = \frac{e^{-\lambda X_l^d}}{e^{1-\lambda_0}}.$$

By using the condition

$$\mathbb{E}[\pi(\omega_l)] = \int \pi(\omega_l)\mathrm{d}\mathbb{P}(\omega_l) = 1$$

we have $\mathbb{E}[e^{-\lambda X_l^d}] = e^{1-\lambda_0}$ or $\lambda_0 = 1 - \log \mathbb{E}[e^{-\lambda X_l^d}]$. Then

$$\pi(\omega_l) = \frac{e^{-\lambda X_l^d}}{\mathbb{E}[e^{-\lambda X_l^d}]}.$$

This expression is equivalent to (11) when $\lambda = p_l$. □

A.6 Proof of Corollary 1

Proof. The proof is similar to that of Theorem 3. A candidate $\bar{\pi}_q$ is the closest belief to the naive belief $1/n_l$ when their Kullback–Leibler divergence is the minimum. The problem is presented as follows:

$$\min_{\bar{\pi}_q} - \sum_{q=1}^{n_l} \bar{\pi}_q \log(n_l \bar{\pi}_q), \ \sum_{q=1}^{n_l} X^d(\omega_l^q) \bar{\pi}_q \geq 0, \sum_{q=1}^{n_l} \bar{\pi}_q = 1$$

where $\sum_{q=1}^{n_l} X^d(\omega_l^q)\bar{\pi}_q \geq 0$ is the positivity constraint as any individual demand is non-negative. The Lagrangian is

$$J(\pi) = \int \sum_{q=1}^{n_l} \bar{\pi}_q \left[\log(n_l \bar{\pi}_q)\right] - \lambda_0 \left(\int \sum_{q=1}^{n_l} \bar{\pi}_q - 1 \right) + \lambda \left(\int \sum_{q=1}^{n_l} X^d(\omega_l^q) \bar{\pi}_q \right)$$

Taking derivative of $J(\bar{\pi}_q)$ and set it to zero, we have $n_l \bar{\pi}_q^* = \exp\left\{\lambda_0 - \frac{1}{n_l} - \lambda X^d(\omega_l^q)\right\}$. Using the condition $\sum_{q=1}^{n_l} \bar{\pi}_q^* = 1$, we have

$$n_l = \exp\left\{\lambda_0 - \frac{1}{n_l}\right\} \times \left(\sum_{q=1}^{n_l} \exp\left\{-\lambda X^d(\omega_l^q)\right\} \right),$$

so $\lambda_0 = \log\left(\sum_{q=1}^{n_l} \exp\left\{-\lambda X^d(\omega_l^q)\right\}/n_l\right) + \frac{1}{n_l}$. It implies

$$\bar{\pi}_q^* = \exp\left\{ -p_l X^d(\omega_l^q) + \sum_q \frac{[e^{-p_l X^d(\omega_l^q)}]}{n_l} \right\}$$

if p_l is the Lagrangian multiplier of the positivity constraint.

Note that the function $e^{-p_l X^d(\omega_l^q)}$ is bounded and continuous, by the weak law of large numbers, there is

$$\frac{\sum_{q=1}^{n_l}[e^{-p_l X^d(\omega_l^q)}]}{n_l} \to \mathbb{E}[e^{-p_l X^d(\omega_l^q)}]$$

as $n_l \to \infty$. Thus the optimal $\bar{\pi}_q^*$ converges to $\pi(\omega_q^l)$ in (10). The result follows. □

A.7 Proof of Theorem 4

The proof is based on some existing results of global optimization. The presentation of the following theorem is modified to fit the current layout.

Theorem. *[10, Theorem 3.3, Theorem 3.6 (i)] If the following conditions are satisfied:*

(i) the equilibrium equations $\mathbf{p}^\top(\mathbf{f}(\mathbf{x})-\mathbf{z}) : \mathbb{R}^K \to \mathbb{R}^+$ is a Lipschitz-continuous function, in particular, $\mathbf{p} \in \mathbb{I}_{\mathbb{C}}(\mathbb{V}(\mathbf{f}, \mathbf{x}^d))$ is a Lipschitz-continuous function of $\mathbf{z}$, and

(ii) the second order gradient $\nabla^2 \left\{\mathbf{p}^\top(\mathbf{f}(-\mathbf{z})\right\}$ with respect to $\mathbf{z}$ is continuous, and

(iii) exists a real number $\epsilon > 0$ such that

$$(\mathbf{z})^\top \left(\nabla \left\{\mathbf{p}^\top(\mathbf{f}-\mathbf{z})\right\}\right) \geq \frac{1+K\epsilon^2}{2} \max\left\{1, \|\nabla \left\{\mathbf{p}^\top(\mathbf{f}-\mathbf{z})\right\}\|\right\}$$

for all $\mathbf{z} \in \mathbb{R}^K$ and $\|\mathbf{z}\|$ is larger than an infinitesimal,

then there exists a vector of stochastic processes $\mathbf{X}^d(t)$ and its probability distribution is

$$\lim_{t\to\infty} \Pi_t = \frac{\exp\left(\mathbf{p}^\top(\mathbf{f}-\mathbf{X}^d)\right) \times \exp(-\epsilon)}{\mathbb{E}\left[\exp\left(\mathbf{p}^\top(\mathbf{f}-\mathbf{X}^d)\right)\right] \times \exp(-\epsilon)}.$$

The process $\mathbf{X}^d(t)$ will converge $\mathbf{x}^d$ with probability one.

Proof. Note that the case $e^{\mathbf{p}^\top(\mathbf{f}-\mathbf{z})} = e^0 = 1$ is a singleton when $\mathbf{z} = \mathbf{x}^d$. When $\mathbf{z}(t)$ is a dynamic vector, $\mathbf{p}(t)$ as a function of $\mathbf{z}(t)$ is also dynamic. The price vector is time varying because $\mathbf{p} \in \mathbb{I}_{\mathbb{C}}(\mathbb{V}(\mathbf{f}, \mathbf{x}^d))$. Let $\mathbf{p}(1) = \mathbf{p}$. If $\mathbf{z}(t)$ is a deterministic function of t, it is known that $e^{\mathbf{p}(t)^\top(\mathbf{f}-\mathbf{z}(t))}$ is the solution of the following ordinary differential equations (ODE)

$$\frac{\mathrm{d}\mathbf{z}(t)}{\mathrm{d}t} = \nabla\left\{\mathbf{p}(t)^\top(\mathbf{f}-\mathbf{z}(t))\right\}$$

where ∇ is the gradient operator with respect to $\mathbf{z}$. The condition that $\nabla\left\{\mathbf{p}(t)^\top(\mathbf{f}-\mathbf{z}(t))\right\}$ is a Lipschitz-continuous function, it ensures that when $\mathbf{z}(t) = \mathbf{x}^d$, the ODE system reaches a stable equilibrium.

Now consider the stochastic version of this ODE system

$$\mathrm{d}\mathbf{X}^d(t) = \nabla_t \left\{\mathbf{p}(t)^\top(\mathbf{f} - \mathbf{X}^d(t))\right\} \mathrm{d}t + \sqrt{2\epsilon}\mathrm{d}\mathbf{B}(t)$$

where $\mathbf{B}(t)$ is a K-vector independent brownian motion. Let $\mathbf{X}^d(1) = \mathbf{X}^d$, then

$$\Pi(\Omega) = \frac{e^{\mathbf{p}^\top(\mathbf{f}-\mathbf{X}^d)}}{\int e^{\mathbf{p}^\top(\mathbf{f}-\mathbf{X}^d)}\mathrm{d}\mathbb{P}(\Omega)}$$

is the distribution of the above stochastic differential equations (SDE). The equilibrium distribution of this SDE is

$$\Pi_t(\Omega) = \frac{\exp\left(\mathbf{p}^\top(\mathbf{f} - \mathbf{X}^d(t))\right) \times \exp(-\epsilon)}{\mathbb{E}\left[\exp\left(\mathbf{p}^\top(\mathbf{f} - \mathbf{X}^d(t))\right)\right] \times \exp(-\epsilon)}.$$

When $t = 1$, $\Pi_t(\Omega) = \Pi(\Omega)$ initiates the SDE. When $t \to \infty$, the vector of processes $\mathbf{X}^d(t)$ degenerates around $\mathbf{x}^d$ with probability one. □

A.8 Proof of Theorem 5

Proof. By Theorem 1, the vectors of prices $\mathbf{h}_1(\mathbf{x}_1)$ and $\mathbf{h}_2(\mathbf{x}_2)$ are available for $\mathbb{V}(\mathbf{f}_1, \mathbf{x}_1^d)$ and $\mathbb{V}(\mathbf{f}_2, \mathbf{x}_2^d)$ respectively.

Note that $f_l(x_1, \dots x_k)$ in $\mathbf{f}_1$ and $f_j(x_{k+1}, \dots, x_K)$ in $\mathbf{f}_2$ both belong to $\mathbb{R}[x_1, \dots x_k, \dots, x_K] = \mathbb{R}[\mathbf{x}]$. The definition of Catersian product gives

$$\mathbb{V}(\mathbf{f}_1, \mathbf{x}_1^d) \times \mathbb{V}(\mathbf{f}_2, \mathbf{x}_2^d) = \big\{\mathbf{x} \in \mathbb{R}^K \mid f_1(x_1, \dots x_k) - x_1^d = \dots = f_k(x_1, \dots x_k) - x_k^d = f_{k+1}(x_{k+1}, \dots, x_K) - x_{k+1}^d = \dots = f_K(x_{k+1}, \dots, x_K) - x_K^d = 0\big\}$$

where $\mathbf{x} = (x_1, \dots x_k, x_{k+1}, \dots, x_K)$. Let

$$\hat{f}_l(x_1, \dots x_{k-m}, x_{k-m+1}, \dots, x_k) = \hat{f}_l(\mathbf{x}) = f_l(x_1, \dots x_k)$$

for $l = 1, \dots, k-m$ so that f_l is explicitly defined as a polynomial in $\mathbf{x}$. Similarly $\hat{f}_j(\mathbf{x}) = f_j(x_{k+1}, \dots, x_K)$ for $j = k+1, \dots, K$. Now there is

$$\mathbb{V}(\mathbf{f}_1, \mathbf{x}_1^d) \times \mathbb{V}(\mathbf{f}_2, \mathbf{x}_2^d) = \Big\{\mathbf{x} \in \mathbb{R}^k \mid \hat{f}_1(\mathbf{x}) - x_1^d = \dots = \hat{f}_k(\mathbf{x}) - x_k^d = \hat{f}_{k+1}(\mathbf{x}) - x_{k+1}^d = \dots = \hat{f}_K(\mathbf{x}) - x_K^d = 0\Big\}$$

where the condition side contains exactly the original polynomials. By the Definition 1, $\mathbb{V}(\mathbf{f}_1, \mathbf{x}_1^d) \times \mathbb{V}(\mathbf{f}_2, \mathbf{x}_2^d) = \mathbb{V}(\mathbf{f}_1, \mathbf{f}_2, \mathbf{x}^d)$ where $\mathbf{x}^d = (x_1^d, \ldots, x_K^d)$. By Theorem 1, the vector of prices $\mathbf{h}(\mathbf{x})$ is available for $\mathbb{V}(\mathbf{f}_1, \mathbf{f}_2, \mathbf{x}^d)$. □

A.9 Proof of Theorem 6

In the theorem (ii), the system of stochastic processes is derived by the use of multidimensional Fokker-Planck equation. Some results are summarized in the following theorem. For details, please refer to [7, Ch. 6.2.2.].

Theorem. *(Multidimensional Fokker-Planck equation [7, Ch. 6.2.2.]) Consider a general Ito diffusion process for the stochastic vector,*

$$\mathrm{d}M_i(t) = a_i(\mathbf{M})\mathrm{d}t + b_{ij}(\mathbf{M})\mathrm{d}W_j(t), \qquad i, j = 1, \ldots, K-1,$$

with drift, $a_i(\mathbf{M})$, diffusion, $b_{ij}(\mathbf{M})$, and the isotropic vector-valued Wiener process, $\mathrm{d}W_j(t)$. The equivalent Fokker-Planck equation for time homogenous joint probability $\phi(\mathbf{M})$ is

$$\frac{\partial \phi}{\partial t} = 0 = -\frac{\partial}{\partial M_i}\big[a_i(\mathbf{M})\phi\big] + \frac{1}{2}\frac{\partial^2}{\partial M_i \partial M_j}\big[B_{ij}(\mathbf{M})\phi\big], \tag{19}$$

with diffusion $B_{ij} = b_{ik}b_{kj}$. A potential solution of (19) exists if

$$\frac{\partial \ln \phi}{\partial M_j} = B_{ij}^{-1}\left(2a_i - \frac{\partial B_{ik}}{\partial M_k}\right) \equiv -\frac{\partial \Lambda}{\partial M_j}, \qquad i, j, k = 1, \ldots, K-1, \tag{20}$$

is satisfied, where Λ satisfies $\phi(\mathbf{M}) = \exp[-\Lambda(\mathbf{M})]$.

Proof. (i) Let $(w_1, \ldots, w_K) \in \mathbb{R}^K$, $z = \sum_{j=1}^K w_j$ and $m_i = w_i/z$ for $i = 2, \ldots, K$ and $m_1 = 1 - \sum_{i=2}^K m_i$. Recall that if $M_i = f(W_i)$ for a continuous transformation $f(\cdot)$, then the density function of $\phi(\mathbf{M})$ and the density $\phi(\mathbf{W})$ follow

$$\phi(\mathbf{M}) = \left|\frac{\partial f(w_1 \ldots w_K)}{\partial(w_1 \cdots w_K)}\right| \phi(\mathbf{W})$$

where $|\partial f(\cdot)/\partial(\cdot)|$ is the determinant of the Jacobian matrix. In our case, $f(w_i) = w_i/z$ for $i = 1, \ldots K$. Then the determinant of the Jacobian matrix of the transformation $w_i \mapsto m_i$ for $i = 1, \ldots, K$ is given by $1/z^{K-1}$ and the joint density $g(w_1, \ldots, w_{K-1}, z)$ of $(W_1, \ldots, W_{K-1}, Z)$, where $Z = \sum_{j=1}^K W_j$. Note $W_i \sim \text{Gamma}(\bar{\alpha}_i, 1)$. The joint density

$g(w_1, \ldots, w_{K-1}, z)$ is given by

$$g(w_1, \ldots, w_{K-1}, z) \overset{(1)}{=} \frac{1}{\Gamma(\bar{\alpha}_1)\cdots\Gamma(\bar{\alpha}_K)} w_1^{\bar{\alpha}_1 - 1} \cdots w_K^{\bar{\alpha}_K - 1} e^{-z} z^{K-1}$$
$$\overset{(2)}{=} \underbrace{\frac{\Gamma(\sum_{j=1}^K \bar{\alpha}_j)}{\Gamma(\bar{\alpha}_1)\cdots\Gamma(\bar{\alpha}_K)} m_1^{\bar{\alpha}_1 - 1} \cdots m_K^{\bar{\alpha}_K - 1}}_{(a)} \underbrace{\frac{1}{\Gamma(\sum_{j=1}^K \bar{\alpha}_j)} z^{\sum_{j=1}^K \bar{\alpha}_j - 1} e^{-z}}_{(b)}.$$

Note that the (b) term is a Gamma$(\sum_{j=1}^K \bar{\alpha}_j, 1)$ and (a) term is a Dirichlet distribution Dir$(\bar{\alpha}_1, \ldots \bar{\alpha}_K)$. The first equality $\overset{(1)}{=}$ comes from the fact that a transformation $w_i \mapsto m_i$ whose density is a product between the inverse determinant of the Jacobian matrix z^{K-1} and a product of Gamma distributions

$$\frac{1}{\Gamma(\bar{\alpha}_1)} w_1^{\bar{\alpha}_1 - 1} e^{-w_1} \times \cdots \times \frac{1}{\Gamma(\bar{\alpha}_1)} w_K^{\bar{\alpha}_1 - 1} e^{-w_K} = \frac{1}{\Gamma(\bar{\alpha}_1)\cdots\Gamma(\bar{\alpha}_K)} w_1^{\bar{\alpha}_1 - 1} \cdots w_K^{\bar{\alpha}_K - 1} e^{-\sum_{j=1}^K w_j}$$

where $\sum_{j=1}^K w_j = z$. The second equality $\overset{(2)}{=}$ comes from the fact

$$w_1^{\bar{\alpha}_1 - 1} \cdots w_K^{\bar{\alpha}_K - 1} = (m_1^{\bar{\alpha}_1 - 1} z^{\bar{\alpha}_1 - 1}) \cdots (m_K^{\bar{\alpha}_K - 1} z^{\bar{\alpha}_K - 1}) = m_1^{\bar{\alpha}_1 - 1} \cdots m_K^{\bar{\alpha}_K - 1} z^{\sum_{j=1}^K \bar{\alpha}_j - K}.$$

Replacing the expression in $\overset{(1)}{=}$ with this result, we have the expression in $\overset{(2)}{=}$. Note that (a) is independent of (b). Thus $(M_1, \ldots, M_K)$ is independent of $Z = \sum_{i=1}^K W_i$.

(ii) By the theorem of the multidimensional Fokker-Planck equation, I show that a scalar potential

$$\Lambda(\mathbf{M}) = -\sum_{i=1}^K (\bar{\alpha}_i - 1) \ln M_i = -\left\{ \sum_{i=2}^K (\bar{\alpha}_i - 1) \ln M_i + (\bar{\alpha}_1 - 1) \ln \left(1 - \sum_{i=2}^K M_i \right) \right\} \quad (21)$$

is the solution of (16). First, it is shown that the coefficients of (16) corresponding to the specifications of (19)

$$a_i(\mathbf{M}) = \frac{1}{2}(\bar{\alpha}_i M_1(t) - \bar{\alpha}_1 M_i(t)),$$
$$B_{ij}(\mathbf{M}) = \begin{cases} M_i M_1, & \text{for } i = j, \\ 0, & \text{for } i \neq j. \end{cases}$$

Second, consider the derivatives in (20) such that

$$-\frac{\partial \Lambda}{\partial M_j} = \frac{\bar{\alpha}_j - 1}{M_j} - \frac{\bar{\alpha}_1 - 1}{1 - \sum_{i=2}^{K} M_i} = \frac{\bar{\alpha}_j - 1}{M_j} - \frac{\bar{\alpha}_1 - 1}{M_1},$$

and

$$B_{ij}^{-1}\left(2a_i - \sum_{k=1}^{K} \frac{\partial B_{ik}}{\partial M_k}\right) = B_{ij}^{-1}\left(\bar{\alpha}_i M_1(t) - \bar{\alpha}_1 M_i(t) - M_1 - M_i\right) = \frac{\bar{\alpha}_j - 1}{M_j} - \frac{\bar{\alpha}_1 - 1}{M_1}.$$

Thus, (20) is satisfied. It means that $\Lambda(\mathbf{M})$ in (21) induces the solution of (16). Finally consider the probability $\phi(\mathbf{M}) = \exp[-\Lambda(\mathbf{M})]$ which is $m_1^{\bar{\alpha}_1 - 1} \cdots m_K^{\bar{\alpha}_K - 1}$. In order to have a probability function, $\phi(\mathbf{M})$ has to satisfy $\int \phi(\mathbf{M}) \mathrm{d}\mathbf{M} = 1$. Note that

$$\int m_1^{\bar{\alpha}_1 - 1} \cdots m_K^{\bar{\alpha}_K - 1} \mathrm{d}\mathbf{M} = \frac{\Gamma(\bar{\alpha}_1) \cdots \Gamma(\bar{\alpha}_K)}{\Gamma(\sum_{j=1}^{K} \bar{\alpha}_j)}.$$

So after normalizing the $m_1^{\bar{\alpha}_1 - 1} \cdots m_K^{\bar{\alpha}_K - 1}$, we have the probability

$$\frac{\Gamma(\sum_{j=1}^{K} \bar{\alpha}_j)}{\Gamma(\bar{\alpha}_1) \cdots \Gamma(\bar{\alpha}_K)} m_1^{\bar{\alpha}_1 - 1} \cdots m_K^{\bar{\alpha}_K - 1}$$

which is the Dirichlet distribution $\mathrm{Dir}(\bar{\alpha}_1, \ldots \bar{\alpha}_K)$. The result follows. □

A.10 Proof of Corollary 2

Proof. Note that when $K \to \infty$ any W_i has a Poisson distribution

$$\lim_{K \to \infty} \text{Gamma}\left(\frac{1}{K}, 1\right) = \frac{x e^{-x}}{\Gamma(1)} = \text{Poi}(1)$$

and $\lim_{K \to \infty} \mathbb{E}\left[\sum_{i=1}^{K} W_i\right] = \lim_{K \to \infty} K \frac{1}{K} = 1$ so $\sum_{i=1}^{\infty} W_i$ has a Gamma$(1,1)$ distribution and it is independent of W_i as the proof in Theorem 6. Then

$$M_i = \frac{W_i}{\sum_{i=1}^{\infty} W_i}$$

follows the Poisson-Dirichlet distribution with parameter 1. On the other hand, when $n \to \infty$, a random partition of the prime factors

$$\left\{\frac{c_1 \log q_1}{\log n}, \dots, \frac{c_K \log q_K}{\log n}\right\}$$

also follows the Poisson-Dirichlet distribution with parameter 1. The result is summarized in [3, Section 2]. As $n = q_1^{\alpha_1} q_2^{\alpha_2} ... q_k^{\alpha_k}$, then the set $\left\{\frac{c_1 \log q_1}{\log n}, \dots, \frac{c_K \log q_K}{\log n}\right\}$ is a partition of 1 such that

$$\sum_{i=1}^{K} \frac{c_i \log q_i}{\log n} = \frac{\log q_1^{c_1} q_2^{c_2} ... q_k^{c_k}}{\log n} = 1.$$

The randomness of this set can be seen as follows. Let $q_i^{c_i} = \tilde{p}_i$. One can randomly enumerate its prime factors $n = \tilde{p}'_1 \tilde{p}'_2 \dots \tilde{p}'_K$ by letting each prime factor $\tilde{p}$ of n be equal to $\tilde{p}'_1$ with probability $\frac{\log \tilde{p}}{\log n}$, then once $\tilde{p}'_1$ is chosen, let each remaining prime factor $\tilde{p}$ of $n/\tilde{p}'_1$ be equal to $\tilde{p}'_2$ with probability $\frac{\log \tilde{p}}{\log n/\tilde{p}'_1}$, and so on and so forth. Thus the set $\left\{\frac{c_1 \log q_1}{\log n}, \dots, \frac{c_K \log q_K}{\log n}\right\}$ is allowed for the random enumeration. The randomized partition of 1 follow the Poisson-Dirichlet distribution with parameter 1. The result follows. □