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Common Ranking and Stability of Overlapping Coalitions

Ana Mauleon* Nils Roehl† Vincent Vannetelbosch‡

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Abstract

Mauleon, Roehl and Vannetelbosch (GEB, 2018) develop a general theoretical framework to study the stability of overlapping coalition settings. Each group possesses a constitution that contains the rules governing both the composition of the group and the conditions needed to leave the group and/or to become a new member of the group. They propose the concept of constitutional stability to study the group structures that are going to emerge at equilibrium in overlapping coalition settings. They combine requirements on constitutions and preferences for guaranteeing both the existence and the emergence of constitutionally stable group structures. In this paper, we show that an alternative way to exclude the ocurrence of closed cycles is to look for constitutions that allow for a common ranking.

Key words: Overlapping Coalitions, Group structures, Constitutions, Stability, Common Ranking.

JEL classification: C72, C78, D85.

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1 Introduction

The understanding of how and why groups form and the precise way in which they affect outcomes of social and economic interactions has been apprehended assuming that each individual can only be member of one of these groups.[1] However, there are many situations in which individuals might be member of more than one group. Free trade agreements are signed among overlapping collections of countries. Joint ventures are formed among overlapping collections of firms. Overlapping groups of individuals may be involved in relationships involving public-goods provision, reciprocity or information-sharing.

Up to now, very little theoretical work exists on overlapping coalition formation settings. In a recent paper, Mauleon, Roehl and Vannetelbosch (2018) provide a general theoretical framework that could be used to study the stability of any situation involving overlapping coalitions, and not only of specific cooperative[2] or non-cooperative games with overlapping coalitions.[3] It departs from previous work in two important aspects. First, it abstracts from activities carried out within each group and supposes that individuals' preferences depend on the group structure. Second, it introduces the notion of constitution to model for each group the rules governing both the composition of the group and the conditions needed to leave the group and/or to become a new member of the group. For instance, some groups could have capacity constraints or some legal requirements regarding the type of member that could be part of the group. In some groups it might be possible to dismiss members but in others there might be a protection against dismissal. Or, in some groups entry might be free but in others it might require the consent of certain number of members (a majority or the unanimity of them, for example). Therefore, both the individuals' preferences and the constitutional design may have a significant impact on the formation and stability of group structures.

Mauleon, Roehl and Vannetelbosch (2018) propose the concept of constitutional stability to predict the group structures that are going to emerge at equilibrium in overlapping coalition settings where the deviating coalition has to take into account the constitution of the group she wants to modify. This concept generalizes previous stability concepts in the literature in which the rules governing the composition of

[1] Ray (2007) and Ray and Vohra (2015) provide surveys of models of coalition formation.

[2] Chalkiadakis et al. (2010) introduce a model for cooperative games with overlapping coalitions that is applicable in situations where agents need to allocate different parts of their resources to simultaneously serve different tasks as members of different coalitions. They explore the stability concept of the core.

[3] See also the work of Myerson (1980), Shenoy and Kraus (1996), Dang et al. (2006), Conconi and Perroni (2002) and Albizuri et al. (2006) for other specific cooperative or non-cooperative models of overlapping coalitions.

each group as well as the exit of current members and/or the arrival of new members were exogenously given[4] or not explicitly considered.[5] The idea of constitutional stability is that modifying the composition of a group (according to its constitution) via the deviation of a feasible deviating coalition needs the consent of both the deviating players and every member of at least one of the supporting coalitions that could grant the admission into the group and/or the departure of the group of such feasible deviating coalition.

Mauleon, Roehl and Vannetelbosch (2018) examine both the existence of constitutionally stable group structures as well as whether the society will reach one of these stable group structures. Following Jackson and Watts (2002), an improving path is defined as a sequence of group structures that can emerge when players join or leave some groups based on the improvement the resulting group structure offers them relative to the current one. Each group structure in the sequence differs from the previous one in that one group is modified by a feasible deviating coalition and every player joining the group strictly prefers the resulting group structure to the current one. Moreover, the deviation should not be blocked and, hence, there should be a supporting coalition that strictly benefits from the deviation. They show that the society induces a constitutionally stable group structure if and only if the constitutions inhibit the occurrence of closed cycles. They then provide requirements on constitutions and individuals' preferences guaranteeing that, from every initial group structure, there always exists an improving path leading to a constitutionally stable group structure. In other words, they show the kind of constitutions that leads to stability and they explore relations between various constitutional arrangements and individuals' preferences guaranteeing stability.

In the present paper, we show that an alternative way to exclude the ocurrence of closed cycles is to look for constitutions that allow for a common ranking (cf. Banerjee et al., 2001; Farrell and Scotchmer, 1988). Our main result states that there are no closed cycles if and only if there exists a common ranking that reflects a certain level of consensus between players. This result extends previous results to the framework of overlapping coalitions, a difficult framework for which there were no general results regarding necessary and sufficient conditions guaranteeing the convergence of the society to a constitutionally stable group structure. We also find that giving more blocking power to the individuals does not necessarily lead to more stability. Although the set of constitutionally stable group structures might

[4] See Jehiel and Scotchmer (2001), Drèze and Greenberg (1980), Caulier et al. (2013a, 2013b), among others.

[5] The most used stability concept in both the traditional non-overlapping scenario and the overlapping one is the core. Typically, it assumes that the deviators only form coalitions among themselves and, thus, no composition and/or admission rules are considered.

become larger, it could happen that the society will never reach one of these stable group structures because higher blocking power might destroy the existence of the common ranking.

The remainder of the paper proceeds as follows. Section 2 introduces the framework of overlapping coalitions and the notion of constitutions. Section 3 defines constitutionally stable group structures and provides an alternative criterion (the existence of a common ranking) for guaranteeing convergence to a constitutionally stable group structure. Finally, Section 4 concludes.

2 The Model

We consider the model of Mauleon, Roehl and Vannetelbosch (2018). Let $N = \{i_1, \ldots, i_n\}$ be a finite set of players and let $M = \{c_1, \ldots, c_m\}$ be a finite set of groups. A group structure h is a mapping $h : M \longrightarrow 2^N$ assigning to each group $c \in M$ a subset of players $h(c) \in 2^N$, with $h(c)$ representing the members of group c.[6] That is, a group structure h indicates which players are members of which groups. Let $\mathcal{H}$ be the set of all group structures and let $h^{\emptyset} \in \mathcal{H}$, with $h^{\emptyset}(c) = \emptyset$ for all $c \in M$, be the empty group structure (i.e. no player is member of any group). The cardinality of $\mathcal{H}$ is $|\mathcal{H}| = 2^{mn}$. For instance, take a society consisting of three players, $N = \{i_1, i_2, i_3\}$ and four groups, $M = \{c_1, c_2, c_3, c_4\}$, where i_1, i_2 and i_3 are members of c_1, i_2 and i_3 are members of c_2 and c_3, and i_1 is the only member of c_4. Then, the group structure h is simply given by

$$h(c) = \begin{cases} \{i_1, i_2, i_3\}, & \text{if } c = c_1 \\ \{i_2, i_3\}, & \text{if } c \in \{c_2, c_3\} \\ \{i_1\}, & \text{if } c = c_4. \end{cases}$$

Each $i \in N$ has rational preferences $\succeq^i$over $\mathcal{H}$. The tuple $\succeq = (\succeq^i)_{i \in N}$ is called a preference profile. Given the preferences, players might have incentives to join or exit some group in a given group structure. We model the changes in the members of a given group by means of the symmetric difference $\pm$ defined by $D' \pm D = (D' \setminus D) \cup (D \setminus D')$ for all $D', D \subseteq N$. Given a group $c \in M$ and a subset of players $D \subseteq N$, let $h \pm (c, D)$ be the group structure that is obtained from $h \in \mathcal{H}$ if the members of c change due to the arrival and/or departure of the players in the deviating coalition D. Players in $D \cap h(c)$ leave the group and players in $D \setminus h(c)$

[6] Note that the tuple (N, M, h) is simply a mathematical hypergraph.

join it.[7] Formally:

$$(h \pm (c, D))(c') := \begin{cases} h(c) \pm D & \text{if } c = c' \\ h(c') & \text{if } c \neq c' \end{cases} \quad (1)$$

If $D \cap h(c) = \emptyset$, we write $h + (c, D)$ instead of $h \pm (c, D)$ meaning that no player leaves the group. If $D \subseteq h(c)$, we write $h - (c, D)$ instead of $h \pm (c, D)$ indicating that no player joins the group.

Each group could have different rules governing the composition of the group as well as the exit of already existing members and/or the arrival of new members. According to these rules, some deviations might not be feasible, given the rules governing the composition of the group. For all $c \in M$ and $h \in \mathcal{H}$, we denote by $\mathcal{D}_h^c \subseteq 2^N \setminus \{\emptyset\}$ the set of feasible deviating coalitions.

Some feasible deviations could be blocked. A feasible deviating coalition $D \in \mathcal{D}_h^c$ would not be blocked only if there is a subgroup of existing members $S \in h(c)$ that agrees with the deviation. For each feasible deviating coalition $D \in \mathcal{D}_h^c$, the set of supporting coalitions that could grant its admission into the group c is denoted by $S_h^c(D) \subseteq 2^{h(c)}$.

A constitution describes both the rules governing composition of the group and the conditions to fulfill in order to be accepted into the group. Formally, the constitution $\mathcal{C}_h^c$ of group $c \in M$ in the group structure $h \in \mathcal{H}$, is a pair $\mathcal{C}_h^c = (\mathcal{D}_h^c, S_h^c)$ where (i) $\mathcal{D}_h^c \subseteq 2^N \setminus \{\emptyset\}$ describes the set of feasible deviating coalitions,[8] and (ii) for each $D \in \mathcal{D}_h^c$, $S_h^c(D) \subseteq 2^{h(c)}$ specifies a non-empty set of supporting coalitions that can grant the admission of the feasible deviating coalition into the group. In addition, if $S \in \mathcal{S}_h^c(D) \setminus \{\emptyset\}$, we assume that $S' \in \mathcal{S}_h^c(D)$ for all $S' \supseteq S$. That is, if S is a supporting coalition for a certain deviating coalition, all coalitions containing S have also the power to support this deviation.

Let $\mathcal{C}^c = (\mathcal{C}_h^c)_{h \in \mathcal{H}}$ be the constitutions of group c in each possible group structure $h \in \mathcal{H}$. Let $\mathcal{C} := (\mathcal{C}^c)_{c \in M}$ be the constitutions of each group in each possible group structure. The tuple $(N, M, \succeq, \mathcal{C})$ is called a society.

[7] We use $\pm$ instead of the usual symbol $\triangle$ for denoting the symmetric difference, in order to emphasize that it might be possible that at the same time new members enter a group while other members leave it.

[8] If $\mathcal{D}_h^c = 2^N \setminus \{\emptyset\}$, then there are no restrictions on feasible deviations in group c. But, if $\mathcal{D}_h^c \subset 2^N \setminus \{\emptyset\}$, then some changes in the composition of the group are not possible.

3 Constitutionally Stable Group Structures

3.1 Existence and common ranking

Mauleon, Roehl and Vannetelbosch (2018) propose the concept of constitutional stability to predict the group structures that one might expect to emerge in the long run when the deviating coalition has to take into account the constitution of the group she wants to deviate. A group structure h' is obtainable from h via the deviating coalition D, $D \subseteq N$, if (i) there is a unique group c whose composition is changed: $h' = h \pm (c, D)$, and (ii) the deviating coalition is feasible according to the constitution of group c: $D \in \mathcal{D}_h^c$.

The group structures that are obtainable from a given group structure h are such that they only differ from h in that a unique group has changed its composition via the deviation of a feasible deviating coalition according to the constitution of that group. A group structure h is constitutionally stable if all coalitional deviations to some obtainable group structure are blocked.

Definition 1. Given the society $(N, M, \succeq, \mathcal{C})$, a group structure h is constitutionally stable with respect to the constitutions $\mathcal{C}$ if for any $D \subseteq N$, $h \pm (c, D)$ obtainable from h via D, we have either $h \succeq^i h \pm (c, D)$ for at least one $i \in D \setminus h(c)$, or in each supporting coalition $S \in \mathcal{S}_h^c(D)$ there exists $j \in S$ such that $h \succeq^j h \pm (c, D)$.

Definition 1 tells us that, under the constitutions $\mathcal{C}$, a group structure $h \in \mathcal{H}$ is constitutionally stable if and only if any feasible deviation of some coalition $D \in \mathcal{D}_h^c$ to some obtainable group structure $h \pm (c, D)$ is deterred because at least one of the deviating players joining c does not strictly benefit from deviating or at least one of the members of every supporting coalition $S \in \mathcal{S}_h^c(D)$ is not strictly better off from the deviation.[9] Some members of the group c may have the power to force other members to leave c even when the excluded players suffer from this exclusion. Hence, moving from $h \in \mathcal{H}$ to $h \pm (c, D)$ does not necessarily need the consent of players leaving c. However, a player who is not in $c \in M$ cannot be forced to join c. Only if she strictly benefits, she will join it.

Let $\mathcal{ST}(\mathcal{C})$ be the set of constitutionally stable group structures with respect to the constitutions $\mathcal{C}$. Generically, constitutionally stable group structures might fail to exist and this leads to the question of how the design of constitutions affects the (non-)existence of stable group structures. Moreover, even if constitutionally stable group structures exist, there is no guarantee that the society will reach one

[9] Under the constitutions $\mathcal{C}$, the move from a group structure h to any obtainable group structure $h \pm (c, D)$ needs the consent of every member of at least one of the supporting coalitions.

of them and not be stucked in a (closed) cycle where a number of group structures are repeatedly visited.

Given $h \in \mathcal{H}$ and $c \in M$, let $\mathcal{A}_h^c(\mathcal{C}) := \{D \in \mathcal{D}_h^c \mid \exists S \in \mathcal{S}_h^c(D)$ such that $h \pm (c, D) \succ^i h \; \forall\, i \in S \cup (D \setminus h(c))$ with $\emptyset \notin S \cup (D \setminus h(c))\}$ be the set of all feasible deviations that are not blocked.[10] That is, the feasible deviations that strictly benefit all the deviating players joining the group and/or all members of at least one of the supporting coalitions.

An improving path is a sequence of group structures that can emerge when players join or leave some groups based on the improvement the resulting group structure offers them relative to the current one. Each group structure in the sequence differs from the previous one in that one group is modified by a feasible deviating coalition and every player joining the group strictly prefers the resulting group structure to the current one. Moreover, the deviation should not be blocked and, hence, there should be a supporting coalition that strictly benefits from the deviation. Formally, an improving path from $h_0 \in \mathcal{H}$ to $h_k \in \mathcal{H}$ is a sequence of group structures $(h_0, h_1, \ldots, h_k)$ such that for all $0 \leq l < k$ we have $h_{l+1} = h_l \pm (c_l, D_l)$ with $c_l \in M$ and $D_l \in \mathcal{A}_{h_l}^{c_l}(\mathcal{C})$. If there exists an improving path from $h \in \mathcal{H}$ to $h' \in \mathcal{H}$, we write $h \mapsto h'$. Moreover, let $I(h) = \{h' \in \mathcal{H} \mid h \mapsto h'\}$ be the set of group structures that can be reached by an improving path starting at h. Notice that h is constitutionally stable if and only if there is no improving path starting at h; that is, $I(h) = \emptyset$.

A set of group structures $H \subseteq \mathcal{H}$ is closed if there is no improving path leading out of it, i.e., $I(h) \subseteq H$ for all $h \in H$. Moreover, a set of group structures $H \subseteq \mathcal{H}$ with $|H| \geq 2$ is a cycle if for any pair $h, h' \in H$, there exists an improving path connecting h to h'.

Lemma 1 (Mauleon, Roehl and Vannetelbosch, 2018)**.** *Let the society $(N, M, \succeq, \mathcal{C})$ be given. There exists no closed cycle if and only if, for each group structure $h \in \mathcal{H}$ that is not constitutionally stable, there is an improving path leading from this group structure to a constitutionally stable one.*

Thus, the non-existence of closed cycles not only implies existence of constitutionally stable group structures but it also guarantees that the society will reach one of these stable group structures. Mauleon, Roehl and Vannetelbosch (2018) provide reasonable restrictions on players' preferences and consistency conditions on the constitutions to guarantee the non-occurrence of closed cycles and the convergence to a constitutionally stable group structure. We show here that an alternative way to exclude the occurrence of closed cycles is to look for constitutions that allow for a common ranking.

[10] We restrict $\emptyset \notin S \cup (D \setminus h(c))$ to guarantee that at least one player strictly benefits from deviating.

Definition 2. Given the society $(N, M, \succeq, \mathcal{C})$, a common ranking $\trianglerighteq$ is a complete and transitive ordering over $\mathcal{H}$ such that $D \in \mathcal{A}_h^c(\mathcal{C})$ implies $h \pm (c, D) \trianglerighteq h$ for all $h \in \mathcal{H}$ and $c \in M$.

A common ranking $\trianglerighteq$ reflects a certain level of consensus between the players in the sense that, whenever a feasible deviation D from h to an obtainable group structure $h \pm (c, D)$ is not blocked, then all players in the society agree that the resulting group structure $h \pm (c, D)$ should be ranked above h. The main idea is that the set of group structures can be decomposed into several equivalence classes and once a higher class is reached, this will not be reversed afterwards. Indeed, a deviation takes place only if the joining and supporting players agree that the resulting group structure is not contained in a lower class than the current one. Note that a priory this is not a restriction at all because it would be possible, for instance, to choose $\trianglerighteq$ in such a way that all group structures are equivalent (i.e., $h \trianglerighteq h'$ as well as $h' \trianglerighteq h$ for all $h, h' \in \mathcal{H}$). This immediately implies that a (not necessarily unique) common ranking always exists. However, the more consensus about beneficial deviations between the players, the stronger the restrictions that can be imposed by a common ranking.

Farrell and Scotchmer (1988) introduce the common ranking property that requires the existence of a linear ordering over all coalitions which coincides with any player's preference ordering over coalitions to which she belongs. A relaxed version of the common ranking property, the top-coalition property, is introduced by Banerjee et al. (2001) to guarantee the existence of a core partition. The common ranking that we introduce here is similar to the one of Farrell and Scotchmer (1988) and orders the group structures differing only in that a unique group c of players has changed its composition according to the preferences of the joining and supporting players involved in the change of c.

Proposition 1. *Let the society $(N, M, \succeq, \mathcal{C})$ be given.*

(i) *There are no cycles if and only if there exists a common ranking $\trianglerighteq$ such that for all $H \subseteq \mathcal{H}$ there is a unique $\trianglerighteq$–maximal group structure $\hat{h} \in H$.*

(ii) *There are no closed cycles if and only if there exists a common ranking $\trianglerighteq$ such that for all $h \in \mathcal{H}$ there is a unique $\trianglerighteq$–maximal group structure $\hat{h} \in I(h)$.*

Proof. **(i)** ($\Rightarrow$) In order to show that existence of $\trianglerighteq$ implies the non-existence of cycles, we will consider the counterposition of this statement. Therefore, assume there is a cycle $H \subseteq \mathcal{H}$. Since there exists a path from each group structure in H to every other group structure in H, if $\trianglerighteq$ is a common ranking, we must have $\bar{h} \trianglerighteq \breve{h}$ as well as $\breve{h} \trianglerighteq \bar{h}$ for all $\bar{h}, \breve{h} \in H$. Thus, there is no unique $\trianglerighteq$-maximal element in H.

($\Leftarrow$) For the other direction suppose there exists no cycle. The following algorithm proceeds in a similar way as the one in the proof of Theorem 1 in Jackson and Watts (2001). We start with the binary relation $\trianglerighteq_1$ where $h \ \triangleright_1 \ \bar{h}$ if and only if there exists an improving path from $\bar{h}$ to h. Because there is no cycle, $\trianglerighteq_1$ is strict. Moreover, for all $h \in \mathcal{H}$, $c \in M$, and $D \in \mathcal{D}_h^c$, deviating from h to $h \pm (c, D)$ always implies $h \pm (c, D) \ \triangleright_1 \ h$ by construction. However, $\trianglerighteq_1$ is not necessarily complete. Let $\tilde{h}, \breve{h} \in \mathcal{H}$ with neither $\tilde{h} \ \triangleright_1 \ \breve{h}$ nor $\breve{h} \ \triangleright_1 \ \tilde{h}$. We construct $\bar{\trianglerighteq}_1$ by adding $\tilde{h} \ \bar{\triangleright}_1 \ \breve{h}$ to $\trianglerighteq_1$, i.e., $h \ \bar{\trianglerighteq}_1 \ \bar{h}$ if and only if $h \ \trianglerighteq_1 \ \bar{h}$ or $h = \tilde{h}$ and $\bar{h} = \breve{h}$. Moreover, let $\trianglerighteq_2$ be the transitive closure of $\bar{\trianglerighteq}_1$. We will show that $\trianglerighteq_2$ still represents the preference profile of the players, i.e., deviating from h to $h \pm (c, D)$ always implies $h \pm (c, D) \ \triangleright_2 \ h$ for all $c \in M$ and $D \in \mathcal{D}_h^c$. Suppose this is not true, that is, suppose there exist $h' \in \mathcal{H}$, $c \in M$, $D \in \mathcal{D}_{h'}^c$, and $S \in \mathcal{S}_{h'}^c$ with $h' \pm (c, D) \succ^i h'$ for all $i \in (D \setminus h'(c)) \cup S$ but $h' \ \trianglerighteq_2 \ h' \pm (c, D)$. Thus, there exists a sequence of group structures $(h_0, h_1, \ldots, h_k)$ with $h_0 = h'$, $h_k = h' \pm (c, D)$ and $h_0 \ \bar{\trianglerighteq}_1 \ h_1 \ \bar{\trianglerighteq}_1 \ \ldots \ \bar{\trianglerighteq}_1 \ h_k$. Assume the sequence is of minimal length. This implies that $h_l = h_{l'}$ only if $l = l'$ for all $l, l' \in \{0, 1, \ldots, k\}$. Suppose there exists an $l \in \{1, \ldots, k\}$ with $\{h_{l-1}, h_l\} = \{\breve{h}, \tilde{h}\}$. Because $h_{l'} \neq \breve{h}, \tilde{h}$ for all $l' \notin \{l-1, l\}$ this yields

$$h_l \ \trianglerighteq_1 \ h_{l+1} \ \trianglerighteq_1 \ \ldots \ \trianglerighteq_1 \ h_k = h' \pm (c, D) \ \trianglerighteq_1 \ h' = h_0 \ \trianglerighteq_1 \ \ldots \ \trianglerighteq_1 h_{l-1}$$

and, thus, there exists an improving path from $\tilde{h}$ to $\breve{h}$ or vice versa. This contradicts the assumption that the two group structures are not comparable under $\trianglerighteq_1$. Therefore, there exists no $l \in \{1, \ldots, k\}$ with $\{h_{l-1}, h_l\} = \{\breve{h}, \tilde{h}\}$. From this follows $h_0 \ \trianglerighteq_1 \ h_1 \ \trianglerighteq_1 \ \ldots \ \trianglerighteq_1 \ h_k$ which contradicts the assumption that there is no cycle. Thus, $\trianglerighteq_2$ still represents the preferences of the players and by construction it is also transitive and strict. If it is not complete, the previous steps can be iterated. Because the set of group structures is finite, the iteration will stop after finitely many steps and we obtain a common ranking $\trianglerighteq$ which is strict. In particular, strictness implies that for each $H \subseteq \mathcal{H}$ there is a unique $\trianglerighteq$-maximal group structure $\hat{h} \in H$.

(ii) ($\Rightarrow$) The first direction proceeds analogously to the first direction of Part (i). Let a common ranking $\trianglerighteq$ and a set of group structures $H \subseteq \mathcal{H}$ be given. If H forms a closed cycle, we have $I(h) = I(h') = H$ and $h \trianglerighteq h'$ as well as $h' \trianglerighteq h$ for all $h, h' \in H$. But this would contradict that there is a unique $\trianglerighteq$-maximal group structure in H and, thus, there cannot exist a closed cycle.

($\Leftarrow$) For the other direction suppose there exist no closed cycles. The first step of the construction of the common ranking proceeds in the same way as the one of Part (i). That is, we start with $\trianglerighteq_1$ where $h \trianglerighteq_1 \bar{h}$ if and only if there exists an improving path from $\bar{h}$ to h. But note that here this binary relation is not necessarily strict. Since by assumption there are no closed cycles, there exists at least one constitutionally stable group structure $h' \in \mathcal{H}$. If this group structure is uniquely determined, according to Lemma 1 it is contained in every closed subset $H \subseteq \mathcal{H}$ and $\trianglerighteq_1$ can then obviously be extended to a complete ranking where h' is the unique maximal element. Therefore, in the following, suppose there exists a further constitutionally stable group structure $h'' \in \mathcal{H}$. In particular, this implies that neither $h' \trianglerighteq_1 h''$ nor $h'' \trianglerighteq_1 h'$. Let $\tilde{h}, \breve{h} \in \mathcal{H}$ be an arbitrary pair of group structures not comparable under $\trianglerighteq_1$. Analogously to above, $\bar{\trianglerighteq}_1$ is constructed by adding $\tilde{h} \;\bar{\triangleright}_1\; \breve{h}$ to $\trianglerighteq_1$, i.e., $h \;\bar{\trianglerighteq}_1\; \bar{h}$ if and only if $h \;\trianglerighteq_1\; \bar{h}$ or $h = \tilde{h}$ and $\bar{h} = \breve{h}$. Again, let $\trianglerighteq_2$ be the transitive closure of $\bar{\trianglerighteq}_1$. Note that by construction $h' \;\trianglerighteq_2\; h''$ would imply $h' \;\triangleright_2\; h''$ and vice versa. If $\trianglerighteq_2$ is not complete, because of finiteness of $\mathcal{H}$ we can iterate the previous steps until a complete ranking $\trianglerighteq$ is reached. We will show that h' and h'' are still not equivalent under $\trianglerighteq$. This, in fact, has the following implication: If $\breve{h}$ is $\trianglerighteq$–maximal in a closed subset $H \subseteq \mathcal{H}$, it has to be constitutionally stable by construction and w.l.o.g. we may assume $\breve{h} = h'$. Then, for any other stable group structure $h'' \in H$, we must have $h' \triangleright h''$ and, thus, h' is the unique $\trianglerighteq$–maximal element in H.

In order to show h' and h'' are still not equivalent under $\trianglerighteq$, let $\trianglerighteq_k$ be the binary relation constructed in the k-th step of the algorithm described in the previous passage. For $k = 1, 2$ we already know that $h' \;\trianglerighteq_k\; h''$ would imply $h' \;\triangleright_k\; h''$ and vice versa. We will show inductively that this is also satisfied for all other k. Therefore, let $k \geq 3$ and suppose that h' and h'' are still not equivalent under $\trianglerighteq_{k-1}$. Moreover, assume this is not satisfied under $\trianglerighteq_k$, i.e., we have $h' \;\trianglerighteq_k\; h''$ as well as $h'' \;\trianglerighteq_k\; h'$. This assumption will lead to a contradiction. Let $\tilde{h}^{(k-1)}, \breve{h}^{(k-1)} \in \mathcal{H}$ be the corresponding pair of group structures not comparable under $\trianglerighteq_{k-1}$ which is added in the next step. We will distinguish three cases:

Case 1: $h' \;\triangleright_{k-1}\; h''$.

Because we assume h' and h'' are not equivalent under $\trianglerighteq_{k-1}$, this implies that there exists a sequence of group structures $(h_1, \ldots, h_l)$ with $h_1 = h''$, $h_l = h'$, and $h_1 \;\bar{\trianglerighteq}_{k-1} \;\ldots\; \bar{\trianglerighteq}_{k-1}\, h_l$. Moreover, from this also follows that there exists $1 \leq l' \leq l-1$ with $\{h_{l'}, h_{l'+1}\} = \{\tilde{h}^{(k-1)}, \breve{h}^{(k-1)}\}$. But then

$$h_{l'+1} \;\trianglerighteq_{k-1}\; \ldots \;\trianglerighteq_{k-1}\; h' \;\triangleright_{k-1}\; h'' \;\trianglerighteq_{k-1}\; \ldots \;\triangleright_{k-1}\; h_{l'},$$

which contradicts that $\tilde{h}^{(k-1)}$ and $\breve{h}^{(k-1)}$ are not comparable under $\triangleright_{k-1}$.

Case 2: $h'' \;\triangleright_{k-1}\; h'$.

This case proceeds analogously to the previous one by just reversing the roles of h' and h''.

Case 3: h' and h'' are not comparable under $\trianglerighteq_{k-1}$.

If h' and h'' are equivalent under $\trianglerighteq_k$ but not under $\trianglerighteq_{k-1}$, there must be two sequences of group structures $(h_1, \ldots, h_l)$ and $(\bar{h}_1, \ldots, \bar{h}_{\bar{l}})$ with $h_1 = \bar{h}_{\bar{l}} = h'$, $h_l = \bar{h}_1 = h''$, and

$$h_1 \trianglerighteq_{k-1} \ldots \trianglerighteq_{k-1} h_l = \bar{h}_1 \trianglerighteq_{k-1} \ldots \trianglerighteq_{k-1} \bar{h}_{\bar{l}}.$$

Moreover, there exist $1 \leq l' \leq l-1$ and $1 \leq \bar{l}' \leq \bar{l}-1$ with $\{h_{l'}, h_{l'+1}\} = \{\bar{h}_{\bar{l}'}, \bar{h}_{\bar{l}'+1}\} = \{\tilde{h}^{(k-1)}, \breve{h}^{(k-1)}\}$. In particular, this yields

$$h_{l'} \trianglerighteq_{k-1} h_{l'+1} \trianglerighteq_{k-1} \ldots \trianglerighteq_{k-1} h'' \trianglerighteq_{k-1} \ldots \trianglerighteq_{k-1} \bar{h}_{\bar{l}'} \trianglerighteq_{k-1} \bar{h}_{\bar{l}'+1}$$

which could only be satisfied if $\tilde{h}^{(k-1)}$ and $\breve{h}^{(k-1)}$ are comparable under $\trianglerighteq_{k-1}$. $\square$

Proposition 1 provides an alternative criterion for guaranteeing convergence to a constitutionally stable group structure. Part (i) states that requiring non-existence of cycles is equivalent to requiring the existence of a special common ranking which identifies a unique maximal element in every subset of group structures. A common ranking meets this requirement if and only if it is strict. In this case, it is a variation of “Generalized Ordinal Potentials” introduced by Monderer and Shapley (1996). In particular, part (i) of Proposition 1 is closely related to their Lemma 2.5. Moreover, it also relates to Theorem 1 in Jackson and Watts (2001). According to part (ii), having this feature only in particular subsets of $\mathcal{H}$ is still strong enough for excluding closed cycles. Therefore, the society induces a constitutionally stable group structure for sure if and only if the constitutions allow for a common ranking which is sufficiently restrictive. That is, there must be some consent about which feasible deviations are beneficial and which are not.

3.2 Blocking power and stability

In our formulation, the constitutions grant the group members a certain level of blocking power allowing them to inhibit changes in the composition of the group that do not conform to their own preferences. Next proposition studies whether enhancing the blocking power of the individuals leads or not to more stability.

Proposition 2. *Let two societies $(N, M, \succeq, \mathcal{C})$ and $(N, M, \succeq, \bar{\mathcal{C}})$ be given and assume that the constitution $\mathcal{C}$ restricts more the set of feasible deviations and the set of supporting coalitions than the constitution $\bar{\mathcal{C}}$, $\mathcal{C} \subseteq \bar{\mathcal{C}}$, i.e., $\mathcal{D}_h^c \subseteq \bar{\mathcal{D}}_h^c$ and $\mathcal{S}_h^c(D) \subseteq \bar{\mathcal{S}}_h^c(D)$ for all $h \in \mathcal{H}$, $c \in M$, and $D \in \mathcal{D}_h^c$. Then, the non-existence of closed cycles under $\bar{\mathcal{C}}$ does not imply that there are no closed cycles under $\mathcal{C}$ even if $\mathcal{ST}(\bar{\mathcal{C}}) \subseteq \mathcal{ST}(\mathcal{C})$.*

Proof. It is sufficient to construct a suitable example. The one we consider here is a variation of an example from Bogomolnaia and Jackson (2002). There are three players $N = \{i_1.i_2, i_3\}$ and one group $M = \{c\}$. Thus, $|\mathcal{H}| = 8$. The group structures are given by:

	$h_1(c)$	$h_2(c)$	$h_3(c)$	$h_4(c)$	$h_5(c)$	$h_6(c)$	$h_7(c)$	$h^\emptyset$
c	$\{i_1\}$	$\{i_2\}$	$\{i_3\}$	$\{i_1, i_2\}$	$\{i_1, i_3\}$	$\{i_2, i_3\}$	$\{i_1, i_2, i_3\}$	$\emptyset$

and the players' preferences are

$$h_4 \succ^{i_1} h_7 \succ^{i_1} h_5 \succ^{i_1} h_1 \succ^{i_1} h_2 \sim^{i_1} h_3 \sim^{i_1} h_6 \sim^{i_1} h^\emptyset$$
$$h_6 \succ^{i_2} h_7 \succ^{i_2} h_4 \succ^{i_2} h_2 \succ^{i_2} h_1 \sim^{i_2} h_3 \sim^{i_2} h_5 \sim^{i_2} h^\emptyset$$
$$h_5 \succ^{i_3} h_7 \succ^{i_3} h_6 \succ^{i_3} h_3 \succ^{i_1} h_1 \sim^{i_3} h_2 \sim^{i_3} h_4 \sim^{i_3} h^\emptyset.$$

The setting is actually not completely the same as in Bogomolnaia and Jackson (2002), because in their paper the authors study coalition formation (i.e., the set of players is always decomposed into a partition) while we have just one group containing some of the players. However, "core stability" in their setting corresponds to constitutional stability with respect to the following constitution $\mathcal{C}^c = (\mathcal{D}^c, \mathcal{S}^c)$:

$$\mathcal{D}^c_h = 2^N \setminus \{\emptyset\} \quad \text{and} \quad \mathcal{S}^c_h(D) = \{S \subseteq h(c) \,|\, (h(c) \setminus D) \subseteq S, S \neq \emptyset\} \tag{2}$$

for all $h \neq h^\emptyset$. Given $\mathcal{C}^c$, a priory all possible deviating coalitions are feasible and a deviation $D \neq h(c)$ takes place if and only if all members of the resulting group structure benefit from the deviation, i.e., $h \pm (c, D) \succ^i h$ for all $i \in h(c) \pm D$. This implies that players who are undesired can be dismissed if the other members of the group agree on this. For the (pathological) special case of $D = h(c)$, it is required that at least one player approves the deviation. Now, given the constitution as defined in (2), we have that h_7 is the unique constitutionally stable (or "core stable", respectively) group structure and $H := \{h_4, h_6, h_5\}$ forms a closed cycle. In fact, once H is reached, there is no improving path leading to h_7 because the players act too myopically. However, consider the following constitutions $\bar{\mathcal{C}}^c = (\bar{\mathcal{D}}^c, \bar{\mathcal{S}}^c)$. Let $\bar{\mathcal{D}}^c_h = 2^N \setminus \{\emptyset\}$ and

$$\bar{\mathcal{S}}^c_h(D) = \begin{cases} \{S \subseteq h(c) \,|\, (h(c) \setminus D) \subseteq S,\, S \neq \emptyset\} & \text{, if } D \cap h(c) \neq \emptyset \\ \{S \subseteq h(c) \,|\, S \neq \emptyset\} & \text{, if } D \cap h(c) = \emptyset \end{cases}$$

for all $h \neq h^\emptyset$. Here, granting access to c to a coalition of players that are not member of the group just needs the support of only one member of the group. This obviously implies $\mathcal{C}^c \subsetneq \bar{\mathcal{C}}^c$ and, thus, the players have less blocking power under $\bar{\mathcal{C}}^c$ than under $\mathcal{C}^c$ (but note that the sets of constitutionally stable group structures coincide). However, under $\bar{\mathcal{C}}^c$, $H = \{h_4, h_6, h_5\}$ does not form a closed cycle any more because for all $h \in H$ there is always one member of c who supports deviating from h to h_7. Therefore, given $\bar{\mathcal{C}}^c$, there exists no closed cycle. $\square$

From the definition of constitutional stability we have that if the sets of feasible deviations and supporting coalitions shrink, the blocking power of each individual increases and the set of constitutionally stable group structures might become larger but never smaller. However, whether more blocking power really implies more stability, strongly depends on the adopted perspective of stability. Although the set of constitutionally stable group structures might become larger the greater the blocking power of the individuals, it could happen that the society will never reach one of these stable group structures because all improving paths leading to them could be destroyed and closed cycles could occur.

4 Conclusion

Mauleon, Roehl and Vannetelbosch (2018) have developed a general theoretical framework in order to study the stability of any overlapping coalition setting, and not only of specific cooperative or non-cooperative games with overlapping coalitions. They have examined both the existence of constitutionally stable group structures as well as whether the society will reach one of these stable group structures. They have shown that the society induces a constitutionally stable group structure if and only if the constitutions inhibit the occurrence of closed cycles. They have provided requirements on constitutions and individuals' preferences guaranteeing that, from every initial group structure, there always exists an improving path leading to a constitutionally stable group structure.

In this paper, we have shown that an alternative way to exclude the ocurrence of closed cycles is to look for constitutions that allow for a common ranking that reflects a certain level of consensus between players. This result extends previous results to the difficult framework of overlapping coalitions and provides a general result regarding necessary and sufficient conditions guaranteeing the convergence of the society to a constitutionally stable group structure. We have also found that giving more blocking power to the individuals does not necessarily lead to more stability.

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