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Approval voting and Shapley ranking

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Abstract

Approval voting allows electors to list any number of candidates and their final scores are obtained by summing the votes cast in their favor. Equal-and-even cumulative voting instead follows the One-person-one-vote principle by endowing each elector with a single vote that may be distributed evenly among several candidates. It corresponds to satisfaction approval voting, introduced by Brams and Kilgour (2014) as an extension of approval voting to a multiwinner election. It also corresponds to the concept of Shapley ranking, introduced by Ginsburgh and Zang (2012) as the Shapley value of a cooperative game with transferable utility. In the present paper, we provide an axiomatic foundation for Shapley ranking and analyze the properties of the resulting social welfare function.

Keywords: approval voting, equal-and-even cumulative voting, ranking game, Shapley value

JEL Classification: D71, C71

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> "Which candidate ought to be elected in a single-member constituency *if all we take into account* is the order in which each of the electors ranks the candidates? ... At the very outset of the argument, we try to move from the *is* to the *ought* and to jump the unbridgeable chasm between the universe of science and that of morals."
>
> (Duncan Black, 1958, p. 55)

Introduction

Approval voting is a method that was studied formally in the 1970s by Weber (1977) and Brams and Fishburn (1978). Given a set of candidates, electors have the possibility of listing any number of candidates whom they consider to be "good for the job." The method simply consists in assigning to each candidate a score equal to the number of electors who have listed that candidate. The winners are those with the largest scores. Beyond being a voting method, rational collective preferences are derived from approval voting as in Borda's method of marks and other scoring methods.

Approval voting has its supporters, starting with Brams and Fishburn, but also its opponents, i.e., Saari and van Newenhizen (1988). Approval voting not been implemented very often, except in some scientific societies such as the American Mathematical Society or the Institute of Electrical and Electronics Engineers. A few other exceptions exist, such as the election of the UN Secretary General. Several experiments have been conducted, in particular by Baujard and Igersheim (2010) following the 2002 and 2007 French presidential elections.

In what follows, we make a distinction between voting, ranking and ordering. Voting is the procedure by which electors submit ballots. Ranking aggregates the electors' choices by assigning a score to each candidate. And a ranking leads to an ordering, which is the ordinal relation on the set of candidates induced by the ranking.

Under approval voting, the number of candidates an elector is allowed to list is not limited a priori and listing additional candidates imposes no direct "cost" on electors. If an elector adds a candidate to her ballot, it has no impact on the scores of the other candidates. Furthermore, electors who list several candidates carry more weight. In that sense, approval voting violates the One-person-one-vote principle often emphasized by the advocates of political equality. That principle is satisfied by equal-and-even cumulative voting (also called block approval voting) whereby an elector's vote is divided evenly among the candidates she lists. For instance, if an elector lists three candidates, each gets 1/3 of a vote instead of 1. Hence, an elector's vote weights less the larger the number of candidates on her ballot and electors have an incentive to limit the sizes of their ballots: Adding a candidate reduces the chances that those already present will be elected. If the objective of an elector is to see elected one of the candidates that she places high in her preferences, she will tend to submit a limited number of candidates among which she is comparatively indifferent. Cumulative voting typically is used in multiwinner elections when electors can spread a fixed number of votes – usually equal to the number of seats to be filled – over one or more candidates.

Electors are assumed to have preferences over candidates. More precisely, we assume that each elector orders the candidates from the most preferred to the least preferred and draws a line somewhere to partition the set of candidates into two sublists, as if her preferences were dichotomous. A ballot reveals that that the candidates above the line are strictly preferred to those below it. Hence, two electors who order the candidates identically may well draw the line at different places. If an elector's preferences are incomplete, the candidates whom she cannot order would just not appear on her ballot. However, even if approval voting is compatible with incomplete preferences, we maintain the assumption of completeness. Furthermore, we leave open the possibility of dichotomous preferences whereby electors are indifferent within and outside their ballots. Under the assumption of dichotomous preferences, Mongin and Maniquet (2015) prove that approval voting induces a non-dictatorial social welfare function that satisfies the Pareto criterion and independence of irrelevant alternatives (IIA). That result does not contradict Arrow's impossibility theorem because assuming dichotomous preferences definitely restricts the domain of individual preferences.

The assumption of dichotomous preferences is, however, far too strong. The candidates listed by an elector are in some sense relatively "close" to each other, but assuming indifference within and outside ballots is not plausible. That assumption is even less plausible for the candidates that an elector does not list. In the present paper, we assume only that electors strictly prefer the candidates they list to those they do not list, an assumption that is an integral part of the definition of a ballot. Notice that, under dichotomous preferences, approval voting is equivalent to Borda's method, which, for each elector, allocates 1 to the candidates listed in her ballot and 0 to the others. The absence of information on electors' preferences is a fundamental difficulty when electors are asked to name candidates without ordering them. It is so for approval voting as well as for any other method that limits the number of candidates an elector can list, including plurality voting.

Equal-and-even cumulative voting corresponds to the concept of Shapley ranking, defined by Ginsburgh and Zang (2012) as the Shapley value of a transferable utility game derived from approval ballots. It also corresponds to the concept of satisfaction approval voting introduced by Brams and Kilgour (2014). Even if those scholars limit that aggregation method to multiwinner elections, nothing actually prevents applying it to single-winner elections. In the present paper, we provide an axiomatic foundation for equal-and-even cumulative voting based on the one-person-one-vote principle. We then move from ranking to ordering and look at the properties of the induced social welfare function.

The paper is organized as follows. and equal-and-even cumulative voting are introduced in Section 1 using the concept of ballot profile that specifies, for each subset of candidates, the number of electors who support it. Section 2 first covers transferable utility games, the Shapley value and its axiomatization. Ranking games are then defined and analyzed. In Section 3, we show that their Shapley values coincide with the ranking derived from equal-and-even cumulative voting. The resulting "Shapley ranking" is then axiomatized in terms of ballot profiles by reference to Shapley's axioms wherein efficiency is translated into the one-man-one-vote principle. Section 4 analyzes the properties of the ordering derived from Shapley ranking, from a social choice perspective. The last section is devoted to concluding remarks.

1. Approval, fractional, plurality and majority voting

Consider a set N of n candidates with $n \geq 2$.[1] There can be any number of electors. Electors have preferences over candidates: $i \succ_h j$ reads "elector h prefers candidate i to candidate j" and $i \sim_h j$ reads "elector h is indifferent between candidates i and j." The weak preference relation $\succsim_h$ represents the preferences of elector h. Preferences are assumed to be complete and rational: $\succsim_h$ is a transitive and reflexive binary relation (a complete preorder) over N. A preference profile specifies a preference ordering for each elector.

1.1 Approval voting

Under approval voting, electors are asked to list the candidates they approve. We denote by $N_h \subset N$ the *approval set* or *ballot* of elector h, that is the set of candidates elector h lists. We assume that $N_h \neq \varnothing$ but do not exclude $N_h = N$. The choice of elector h can be identified to an n-tuple $q_h \in \{0,1\}^n$ with $q_{ih} = 1$ if and only if $i \in N_h$. M denotes the set of electors, a ballot profile can be arranged in a $n \times m$ matrix whose rows are attributed to candidates and columns to electors. A ballot profile is a mapping π that associates to each subset $S \subset N$ of candidates, the number of electors whose approval set coincides with S:

$$\pi(S) = \left| \left\{ h \in M \mid N_h = S \right\} \right| \text{ for all } S \subset N. \tag{1}$$

A one-to-one relation exists between ballot profiles and its matrix representation, knowing that the number of electors is obtained by summing the $\pi(S)$. set of admissible ballot profiles on a set N of candidates is given by

$$\Pi(N) = \left\{ \pi \in \mathbb{N}^{2^n - 1} / 0 \middle| \pi(S) \leq n \text{ for all } S \subset N, S \neq \varnothing \right\},$$

where $\mathbb{N} = \{0,1,2,...\}$ denotes the set of nonnegative integers. Notice that the profile (0, 0, ..., 0) is excluded because electors are assumed to submit nonempty ballots.

[1] *Notation*: Upper-case letters are used to denote finite sets and subsets, and the corresponding lower-case letters are used to denote the number of their elements: $n = |N|$, $s = |S|$,...

Example 1 Consider the voting situation with four candidates described by the following ballot profile: [2]

$$\pi(1)=\pi(1,2)=\pi(2,3)=\pi(3,4)=\pi(2,3,4)=1,$$

$$\pi(2)=\pi(3)=\pi(4)=\pi(1,3)=\pi(1,4)=\pi(2,4)=0,$$

$$\pi(1,2,3)=\pi(1,2,4)=\pi(1,3,4)=\pi(1,2,3,4)=0.$$

There are five electors and their approval sets are $N_1=\{1\}$, $N_2=\{1,2\}$, $N_3=\{2,3\}$, $N_4=\{3,4\}$ and $N_5=\{2,3,4\}$. The associated matrix Q is given by:

$$Q=\begin{bmatrix}1&1&0&0&0\\0&1&1&0&1\\0&0&1&1&1\\0&0&0&1&1\end{bmatrix}$$

The *approval score* of candidate *i* is the number of electors who have listed him:

$$AR_i(N,\pi)=\left|\{h\in M \mid i\in N_h\}\right|=\sum_{S:i\in S}\pi(S) \quad (i=1,\ldots,n). \tag{2}$$

It is obtained by summing along each row of the representative matrix Q. In Example 1, the approval ranking is (2, 3, 3, 2). It leads to the following ordering $2\sim 3\succ 1\sim 4$.

1.2 Equal-and-even cumulative voting

Under equal-and-even cumulative voting, each of the candidates listed by elector h receives a fraction $1/n_h$, where n_h is the size of elector h's ballot. The scores are obtained by summing the fractions allocated to each candidate

$$SR_i(N,\pi)=\sum_{S:i\in S}\frac{1}{s}\pi(S)=\sum_{h:i\in N_h}\frac{1}{n_h} \quad (i=1,\ldots,n). \tag{3}$$

Equation (3) coincides with the concept of Shapley ranking introduced by Ginsburgh and Zang (2012) and with the concept of satisfaction approval voting introduced by Brams and Kilgour (2014).Table 1 shows the approval scores (*AR*) and Shapley scores (*SR*) obtained in Example 1.

	1	**2**	**3**	**4**	**5**	*AR*	*SR*
1	1	1	0	0	0	2	1.50
2	0	1	1	0	1	3	1.33
3	0	0	1	1	1	3	1.33
4	0	0	0	1	1	2	0.83

Table 1: Approval and Shapley scores of Example 1

[2] When there is no ambiguity, braces will be omitted.

The resulting ordering $1 \succ 2 \sim 3 \succ 4$ places candidate 1 on top and, not surprisingly, it differs from the ordering $2 \sim 3 \succ 1 \sim 4$ that results from approval ranking. Approval and Shapley orderings obviously coincide in the case of two candidates. Example 1 shows that they may not coincide beyond two candidates. The following example illustrates a situation wherein the two orderings coincide.

Example 2 Consider a voting situation involving three candidates and five electors whose approval sets are $N_1 = \{1\}$, $N_2 = \{1,2\}$, $N_3 = \{2,3\}$, $N_4 = \{1,3\}$ and $N_5 = \{1,2,3\}$. Approval and Shapley scores are given in Table 2. In both cases, candidate 1 comes first while the other two obtain the same score: $1 \succ 2 \sim 3$.

	1	**2**	**3**	**4**	**5**	*AR*	*SR*
1	1	1	0	1	1	4	2.33
2	0	1	1	0	1	3	1.33
3	0	0	1	1	1	3	1.33

Table 2: Approval and Shapley scores of Example 2

Notice that by normalizing Shapley scores, we obtain the probabilities that a particular candidate is elected if the *random dictator* procedure were adopted.[3] Each of the *m* electors is asked to identify a subset of candidates, knowing that a elector will first be chosen at random and the winner will then be chosen at random in her approval set. The resulting probabilities are then proportional to Shapley scores:

$$\text{Prob}[\, i \text{ is elected}\,] = \frac{1}{m} \sum_{h:\, i \in N_h} \frac{1}{n_h} = \frac{1}{m} SR_i(N, \pi).$$

In Example 1 (see Table 1), the probabilities are given by (0.30, 0.27, 0.27, 0.17). In example 3 (see Table 2), the probabilities are given by (0.47, 0.27, 0.27).

1.3 Plurality and majority voting

A number of voting rules in which electors are allowed to submit only one candidate are special cases of approval voting. Those are the cases of plurality and majority voting. The two methods are well defined only in the absence of indifference in individual preferences. Each elector has then a unique most preferred candidate and candidates are ordered according to their approval scores given by (1) or, equivalently, by (2). In plurality voting, the winners are the candidates with the largest approval score. In majority voting, the winner is the candidate with an approval score exceeding half the number of electors. The latter therefore is not a decisive method. The following example shows that a candidate who appears first in a majority of electors' preferences may be defeated under approval voting and equal-and-even cumulative voting. It

[3] This is the terminology used by Bogolmania et al. (2005).

illustrates how voting by approval reveals some information on electors' intensities of preferences.

Example 3 Consider a voting situation with four candidates and five electors whose preferences are given by:

$$1 \succ_1 3 \succ_1 2 \succ_1 4,\ 1 \succ_2 2 \succ_2 3 \succ_2 4,\ 1 \succ_3 2 \succ_3 4 \succ_3 3, 2 \succ_4 3 \succ_4 4 \succ_4 1 \text{ and } 2 \succ_5 4 \succ_5 1 \succ_5 3.$$

The first candidate has a majority: he comes on top of 3 out of 5 orderings.[4] Now assume that the electors submit the following ballots: $N_1 = \{1,3\}$, $N_2 = \{1,2\}$, $N_3 = \{1,2,4\}$, $N_4 = \{2,3\}$ and $N_5 = \{2,4\}$. Table 3 shows the approval and Shapley scores. Candidate 2 gets the largest approval score as well as the largest Shapley score.

	1	**2**	**3**	**4**	**5**	*AR*	*SR*
1	1	1	1	0	0	3	1.33
2	0	1	1	1	1	4	1.83
3	1	0	0	1	0	2	1
4	0	0	1	0	1	2	0.83

Table 3: Approval and Shapley scores of Example 3

When indifference between candidates is not ruled out, plurality and majority voting are not well defined because electors may have several most preferred candidates. If this is the case, electors have to make a selection. We could assume that the name a elector submits is drawn at random among her top candidates. Denoting by N_h the subset of most preferred candidates of elector *h*, each one is assigned a probability equal to $1/n_h$ and the score of a candidate is given by the sum of the probabilities that his name is submitted. In this case, plurality voting and Shapley voting give rise to the same result.

2. Ranking games and their Shapley value

Before defining ranking games, we first recall the definition of a transferable utility game and various related concepts, including the Shapley value.

2.1 Transferable utility games

A *cooperative game with side payments* is defined by a finite set *N* of *n* players and a function *v* (called *characteristic function*) that associates a real number to each subset of players. $v(S)$ is referred as the "worth" of coalition *S*. It could be a surplus or a cost that measures what a coalition can generate without the participation of the other players. $v(N)$ measures what the "grand coalition" can generate when all players best coordinate themselves.[5] By convention,

[4] He is therefore also the unique Condorcet winner (see Section 4).

[5] Games with transferable utility and the notion of characteristic function have been introduced by John von Neumann and Oskar Morgenstern in their 1944 book.

$v(\emptyset) = 0.$ Because a characteristic function on N is defined by a list of $2^n - 1$ real numbers, the set $G(N)$ of all characteristic functions on N can be identified to $\mathbb{R}^{2^n-1}$ the real vector space of dimension $2^n - 1$. The *dual* of a characteristic function v is defined by

$$v^*(S) = v(N) - v(N \setminus S) \text{ for all } S \subset N. \tag{4}$$

$v^*(S)$ is the contribution of coalition S to the grand coalition.[6] Alternatively, it is what remains for coalition S once the complementary coalition has collected its worth. Given a subset $T \subset N$, the unanimity game (N, u_T) is defined by:

$$u_T(S) = \begin{cases} 1 & \text{if } T \subset S, \\ 0 & \text{otherwise.} \end{cases}$$

The $2^n - 1$ unanimity games form a basis of the vector space $G(N)$. Given any characteristic function v on N, there exists a *unique* collection $(\alpha_T \mid T \subset N, T \neq \emptyset)$ of $2^n - 1$ real numbers such that:

$$v(S) = \sum_{T \subset N} \alpha_T(N,v) u_T(S) = \sum_{T \subset S} \alpha_T(N,v) \text{ for all } S \subset N. \tag{5}$$

The coefficients α_T are known as the *Harsanyi dividends* (dividends for short).[7] They are obtained from the following recursive formula deduced from (5), starting with $\alpha_\emptyset = 0$: [8]

$$\alpha_T(N,v) = v(T) - \sum_{S \subsetneq T} \alpha_S(N,v) \text{ for all } T \subset N. \tag{6}$$

As a consequence, there is a one-to-one relation between games and dividends.

For a given game (N,v), the *marginal contribution* of player i to a coalition S is defined by $v(S) - v(S \setminus i)$. Players i and j are substitutable in a game (N,v) if their marginal contributions to coalitions to which they both belong are identical. Equivalently, i and j are substitutable if:

$$v(S \setminus i) = v(S \setminus j) \text{ for all } S \subset N \text{ such that } i, j \in S.$$

Obviously, players who are substitutable in a game are substitutable in the dual. Using (6), it is easily verified that players i and j are substitutable if $\alpha_{S \setminus i}(N,v) = \alpha_{S \setminus j}(N,v)$ for all $S \subset N$ such that $i, j \in S$. Player i is *null* in a game (N,v) if his marginal contributions are all equal to zero: $v(S) = v(S \setminus i)$ for all $S \subset N$. Player i is *dummy* in a game (N,v) if he systematically contributes his worth: $v(S) - v(S \setminus i) = v(i)$ for all $S \subset N$ such that $i \in S$. A null (resp. dummy) player in a game is null (resp. dummy) in the dual. Player i is null if $\alpha_T(N,v) = 0$ for all $T \ni i$.

[6] It is easily verified that $(v^*)^* = v$.

[7] See Harsanyi (1959).

[8] Notation: The symbol $\subsetneq$ is used to denote a strict inclusion.

The *sum* of two games (N, v_1) and (N, v_2) on a common set of players is the game (N, w) defined by $w(S) = v_1(S) + v_2(S)$. The resulting dividends are equal to the sum of the dividends associated with the two games.

2.2 The Shapley value

The central question addressed by the theory of cooperative games is the allocation among players of the amount resulting from their cooperation. A *solution* of a game (N, v) specifies a payoff x_i for each player *i* and an *allocation rule* is a mapping $\varphi : G(N) \to \mathbb{R}^n$ that associates solutions to games. The Shapley value is the rule that allocates uniformly the dividends of every coalition to its members:

$$SV_i(N,v) = \sum_{T:i\in T} \frac{1}{t}\alpha_T(N,v) \quad (i = 1,..., n). \tag{7}$$

Shapley (1953) proves that there is a unique allocation rule $\varphi : G(N) \to \mathbb{R}^n$ that satisfies the following four axioms.

Efficiency: The worth of the grand coalition is exactly allocated, not more nor less.

Anonymity: If players' names are permuted, the solution should be permuted accordingly.

Null player: A zero amount is allocated to null players.

Additivity: The value of a sum of games is the sum of the values.

Anonymity is sometimes replaced by Symmetry, an axiom requiring that an identical amount should be allocated to players who are substitutable. The Null player axiom can be replaced by the Dummy player axiom that requires that dummy players get their stand alone worth. The Shapley value has many other properties. In particular, it is a *self-dual* concept. The value of a game coincides with the value of its dual: $SV(N, v^*) = SV(N, v)$.

2.3 Ranking games and their duals

In a voting context, the candidates are the players. Given a ballot profile (N, π) as defined in (1), we look for a definition of a characteristic function *v* that is an equivalent representation such that $v(N)$ equals the number of electors. To each subset *S* of candidates, we associate the number of electors whose approval set is *included* in *S* :

$$\underline{v}(S) = \left| \left\{ h \in M \mid N_h \subset S \right\} \right| \text{ for all } S \subset N. \tag{8}$$

The function $\underline{v}$ defines a cooperative game with side payments $(N, \underline{v})$ such that $\underline{v}(N) = m$ and $\underline{v}(i)$ is the number of electors who have only listed candidate *i*: $N_h = \{i\}$. It is the concept of *ranking game* introduced by Ginsburgh and Zang (2012). A solution of a ranking game provides a ranking of candidates by specifying for each of them a score equal to a fraction of the total number of electors. The characteristic function associated to Example 1 is given by:

$$\underline{v}(1)=1,\ \underline{v}(2)=0,\ \underline{v}(3)=0,\ \underline{v}(4)=0,$$
$$\underline{v}(1,2)=2,\ \underline{v}(1,3)=1,\ \underline{v}(1,4)=1,\ \underline{v}(2,3)=1,\ \underline{v}(2,4)=0,\ \underline{v}(3,4)=1,$$
$$\underline{v}(1,2,3)=3,\ \underline{v}(1,2,4)=2,\ \underline{v}(1,3,4)=2,\ \underline{v}(2,3,4)=3,$$
$$\underline{v}(1,2,3,4)=5.$$

We observe that a ranking game can be written as the sum of the unanimity games associated with electors:

$$\underline{v}(S)=\sum\nolimits_h u_{N_h}(S) \text{ for all } S\subset N. \tag{9}$$

where the game (N,u_{N_h}) can be viewed as the ranking game associated to elector *h*. (8) can alternatively be written in terms of the ballot profile:

$$\underline{v}(S)=\sum_{T\subset S}\pi(T) \text{ for all } S\subset N. \tag{10}$$

Following (5) and (10), there is a one-to-one relationship between ballot profiles and ranking games.

Proposition 1 The dividends of the ranking game $(N,\underline{v})$ coincide with its underlying ballot profiles: $\alpha_T(N,\underline{v})=\pi(T)$ for all $T\subset N$.

The subset $RG(N)\subset G(N)$ of all ranking games on a set N generated by the set of ballot profiles $\Pi(N)$ forms a remarkable class of games. The characteristic functions defining ranking games are *monotonic* (increasing) and take values in $\mathbb{N}$. They are sums of unanimity games and, because unanimity games are convex, they are *convex*. [9] Ranking games are *positive* in the sense that their dividends are nonnegative.[10] Furthermore, the set $RG(N)$ is *closed under addition*. Starting from any two voting situations (N,π') and (N,π'') on a common set of candidates, and their associated ranking games $(N,\underline{v}')$ and $(N,\underline{v}'')$, the ranking game $(N,\underline{v}'+\underline{v}'')$ is associated to the voting situation $(N,\pi'+\pi'')$. It is implicitly assumed that the sets of electors are disjoint.

Counting the number of electors whose approval set *intersects* S leads to another characteristic function:

$$\overline{v}(S)=\left|\left\{h\in M \mid N_h\cap S\neq\varnothing\right\}\right| \text{ for all } S\subset N. \tag{11}$$

It is such that $\overline{v}(N)=m$ and $\overline{v}(i)$ equals the number of electors who have included candidate i in their approval set i.e. $\overline{v}(i)=AV_i$ as defined by (2). The characteristic functions $\underline{v}$ associated to Example 1 is given by:

[9] Convex games have been introduced and studied by Shapley (1971).

[10] Positive games form a particular subclass of convex games on which the set of asymmetric values obtained by considering all distributions of dividends (the "Harsanyi set") coincides with the set of weighted Shapley values and the core. See Dehez (2017) for details.

$\overline{v}(1)=2, \overline{v}(2)=3, \overline{v}(3)=3, \overline{v}(4)=2,$

$\overline{v}(1,2)=4, \overline{v}(1,3)=5, \overline{v}(1,4)=4, \overline{v}(2,3)=4, \overline{v}(2,4)=4, \overline{v}(3,4)=3,$

$\overline{v}(1,2,3)=5, \overline{v}(1,2,4)=5, \overline{v}(1,3,4)=5, \overline{v}(2,3,4)=4,$

$\overline{v}(1,2,3,4)=5.$

The following proposition states that there is a one-to-one relationship between the games $(N,\underline{v})$ and $(N,\overline{v})$. Comparing the two games, we observe that $\underline{v}(S)\le\overline{v}(S)$ for all $S\subset N$. While $\underline{v}(S)$ is the number of electors who are *exclusively* supporting some candidates in *S*, $\overline{v}(S)$ is the number of electors who are supporting *some* candidates in *S* and it can be seen as the approval score of coalition *S*.

Proposition 2 The games $(N,\underline{v})$ and $(N,\overline{v})$ whose characteristic functions are defined by (8) and (11) are *dual* of each other: $\overline{v}=\underline{v}^*$ and $\underline{v}=\overline{v}^*$.

Proof. To show that $\underline{v}=\overline{v}^*$, consider some subset *S*. The two sets $\left\{h\in M \mid N_h\subset S\right\}$ and $\left\{h\in M \mid N_h\not\subset S\right\}$ form a partition of *M* where

$$\left\{h\in M \mid N_h\not\subset S\right\}=\left\{h\in M \mid N_h\cap(N\setminus S)\neq\varnothing\right\}.$$

Hence, we have $\left|\left\{h\in M \mid N_h\subset S\right\}\right|=m-\left|\left\{h\in M \mid N_h\cap(N\setminus S)\neq\varnothing\right\}\right|$. The proposition then follows from the definition of dual games (4). ♦

As a consequence, a ballot profile (N,π) can be described in two *equivalent* ways, by a ranking game $(N,\underline{v})$ or by its dual $(N,\overline{v})$. The two characteristic functions coincide for $n=2$. Considering the two extreme voting situations where electors list either a single candidate or all candidates, the ranking game and its dual coincide in the first situation while in the second situation, $\overline{v}(S)=m$ for all $S\subset N$ $(S\neq\varnothing)$, and $\underline{v}(S)=0$ for all $S\neq N$.

3. Shapley ranking

We first confirm that equal-and-even cumulative voting associated to a ballot profile coincides with the Shapley value of the corresponding ranking game. We then proceed with an axiomatization expressed directly in terms of ballot profiles.

3.1 The Shapley value of a ranking game

Using the axioms that underlie the Shapley value and the additive decomposition of ranking games given by (9), Ginsburgh and Zang (2012) prove that the Shapley value of a ranking game coincides with equal-and-even cumulative voting as defined by (3). This is confirmed by (7) together with Proposition 1.

Proposition 3 The Shapley ranking associated to a ballot profile (N,π) is the (common) Shapley value of the corresponding dual games $(N,\overline{v})$ and $(N,\underline{v})$:

$$SR_i(N,\pi)=SV_i(N,\overline{v})=SV_i(N,\underline{v}) \quad (i=1,\ldots,n). \tag{12}$$

Proof. The Shapley value is additive. Hence, using (9), the Shapley value of the game $(N, \underline{v})$ is the sum of the values of unanimity games (N, u_{N_h}). Candidates outside N_h receive zero by the Null player axiom and candidates inside N_h receive an equal amount by the axiom of Anonymity. Hence, by the axiom of Efficiency, we have:

$$
\begin{aligned}
SV_i(N, u_{N_h}) &= \frac{1}{n_h} \quad \text{if } i \in N_h, \\
&= 0 \quad \text{if } i \notin N_h.
\end{aligned}
$$

Adding these values results in (3) and (12) follows by self-duality. ♦

3.2 Axiomatization of Shapley ranking

For a given set *N*, we denote by *P*(*N*) the set of *permutations* of *N*. For a given subset $S \subset N$, *pS* denotes the image of *S* under the permutation $p \in P(N)$. For a given set function *v* on *N*, the function *pv* is defined by $pv(pS) = v(S)$ for all $S \subset N$. The following axioms are the translations of Shapley's axioms in terms of ballot profiles. They apply to the ranking rules $\varphi : \Pi(N) \to \mathbb{R}^n$ defined on the set of ballot profiles.

One-person-one-vote (Efficiency) The scores add-up to the number of electors:

$$\sum_{i \in N} \varphi_i(N, \pi) = \sum_{S \subset N} \pi(S).$$

Neutrality (Anonymity) If candidates' names are permuted, scores are permuted accordingly:

For all $p \in P(N)$ and $i \in N$, $\varphi_{p\{i\}}(N, p\pi) = \varphi_i(N, \pi)$.

Null candidate (Null player) Candidates appearing in no ballot get a zero score:

$\pi(S) = 0$ for all $S \subset N$ such that $i \in S \;\Rightarrow\; \varphi_i(N, \pi) = 0$.

Additivity The score associated with a sum of ballot profiles on a common set of candidates is equal the sum of the scores associated with each ballot profile: for any two ballot profiles π' and π'' on *N*, $\varphi(N, \pi' + \pi'') = \varphi(N, \pi') + \varphi(N, \pi'')$.

Additivity makes sense here because the set $\Pi(N)$ of all possible ballot profiles is closed under addition. These four axioms are natural requirements and characterize Shapley ranking.

Proposition 4 Shapley ranking is the unique ranking rule $\varphi : \Pi(N) \to \mathbb{R}^n$ that satisfies One-person-one-vote, Neutrality, Null candidate and Additivity.

Proof Shapley ranking obviously satisfies the four axioms. Now, consider a ranking rule φ satisfying all four axioms. Any ballot profile π on *N* can be decomposed as a sum of elementary ballot profiles $\pi = \sum \pi_T$ where

$$
\begin{aligned}
\pi_T(S) &= \pi(T) \quad \text{if } S = T, \\
&= 0 \quad \text{if } S \neq T.
\end{aligned}
$$

By the Null candidate axiom, we have:

$$\varphi(N,\pi_T)=0 \text{ for all } i\notin T$$

and by the Neutrality axiom, we have:

$$\varphi(N,\pi_T)=\frac{\pi(T)}{t} \text{ for all } i\in T.$$

We then recover Shapley ranking using Additivity. ♦

It is easily seen that approval ranking obtains by replacing the One-person-one-vote axiom by an axiom specifying that the scores add-up to the number of votes (One-person-many-votes).

4. From ranking to ordering

Approval and Shapley orderings are derived from approval and Shapley rankings and they generally differ, as in Example 1. They coincide in the two extreme voting situations where either $n_h=1$ for all *h* or $n_h=n$ for all *h*. In the first situation, analogous to plurality voting, the game $(N,\bar{v})$ is additive and $AR_i=SR_i$ for all *i*. In the second situation, $AR_i=m$ and $SR_i=m/n$ for all *i* where *m* is the number of electors.

Referring to the underlying individual preferences, approval and Shapley orderings are two social welfare functions that assign collective preferences to individual preferences. What are their properties and how do they compare to each other? Not surprisingly, we will see that little can be said outside the case of dichotomous preferences.

4.1 Individual preferences

Concerning individual preferences, several assumptions are possible:

A1 $i\in N_h$ and $j\notin N_h$ implies $i\succ_h j$.

A2 $i,j\in N_h$ implies $i\sim_h j$.

A3 $i,j\in N\setminus N_h$ implies $i\sim_h j$.

A1 is part of the definition of approval voting. The other two assumptions are less natural and far too restrictive, especially A3. The three assumptions together characterize *dichotomous preferences*. Electors are then indifferent between candidates within their approval sets as well as outside.

4.2 From individual to collective preferences

The validity of four axioms will be considered, *Pareto*, *Independence of irrelevant alternatives*, *Condorcet* and *Monotonicity*, on the basis of the approval sets $(N_1,\dots,N_m)$ submitted by electors.

The Pareto principle requires that unanimity should be reflected in collective preferences. It requires that if a candidate *i* is preferred to candidate *j* by all electors, then *i* must be collectively preferred to *j*. If preferences are dichotomous, candidate *i* is preferred to candidate *j* by all electors if and only if $i \in N_h$ and $j \notin N_h$ for all $h \in M$. Clearly, approval and Shapley ranking both satisfy the Pareto principle under dichotomous preferences. Assuming only A1 requires a modified version of the Pareto principle that allows for a situation where *i* and *j* both end up on top of the collective preferences.

Pareto principle If candidate *i* is preferred to candidate *j* by all electors, then *j* cannot be collectively preferred to *i*.

Proposition 5 Under assumption A1, approval and Shapley orderings satisfy the modified Pareto principle.

Proof Consider two candidates *i* and *j* such that $i \succ_h j$ for all $h \in M$. For each elector *h*, there are three cases that define a partition of the set of electors:

$i \in N_h$ and $j \notin N_h \;\rightarrow\; h \in M_1$,

$i, j \in N_h \;\rightarrow\; h \in M_2$,

$i, j \notin N_h \;\rightarrow\; h \in M \setminus (M_1 \cup M_2)$.

The difference in approval scores is then equal to $m_1 \geq 0$ and nothing excludes a situation where M_1 is empty. The difference in Shapley scores is non-negative as well. Indeed, referring to the representative matrix *Q* that describes electors' ballots, we have:

$$SR_i - SR_j = \sum_{h \in M_1} \frac{q_{hi}}{b_h}$$

where $b_h = \sum_{l \in N} q_{hl} > 0$ for all $h \in M$ by assumption. ♦

Arrow (1951) introduced the axiom of independence of irrelevant alternatives (IIA for short), a property that appears to be a natural requirement although it is often not satisfied in the absence of restrictions on individual preferences. Consider two preference profiles with a common set of candidates and a common set of electors, and two candidates *i* and *j*.

IIA If electors have the same preferences regarding *i* and *j* in both profiles, the collective preferences regarding *i* and *j* derived from the two profiles must be identical.

Arrow's impossibility theorem states that, without restrictions on preferences (axiom of *Unrestricted domain*), dictatorship is the only social welfare function that satisfies the Pareto principle and IIA. It is easy to show that, under dichotomous preferences, approval voting satisfies IIA.[11] Indeed, consider two ballot profiles $(N_1, ..., N'_m)$ and $(N''_1, ..., N''_m)$, and the associated representative matrices Q' and Q''. If electors have the same preferences regarding *i* and *j*, the rows *i* and *j* of the matrices Q' and Q'' are identical and therefore $AR'_i = AR''_i$ and

[11] This is acknowledged by Brams and Fishburn (2007, p.137) and confirmed by Maniquet and Mongin (2015).

$AR'_j = AR''_j$. Maniquet and Mongin (2015) show however that approval ordering violates IIA if the *range* is forced to be dichotomous as well.

Shapley ranking instead does not satisfy IIA, whether preferences are dichotomous or not, because of the strong interdependence that characterizes it, as the following example confirms.

Example 5 Consider two ballot profiles on a set of three candidates and five electors, represented by the matrices Q' and Q''. Assume that there are five electors who have the same preferences regarding candidates 1 and 2.

$Q' =$

1	1	1	0	0
0	0	1	1	1
1	1	1	0	0

$Q'' =$

1	1	1	0	0
0	0	1	1	1
0	0	1	1	1

In Q', candidate 2 has a higher Shapley score than candidate 1. The order is reversed in Q''.

The Condorcet principle is often considered to be desirable property. A candidate is a *Condorcet winner* if he never loses in pairwise contests. There may be no Condorcet winner and, if such a candidate exists, one could argue that he should be elected.

Condorcet principle Condorcet winners, if any, should be on top of the collective preferences.

Few aggregation methods satisfies this principle. Of course, if individual preferences were known (like in scoring methods), one could first check whether there is a Condorcet winner and elect him if he exists. If preferences are assumed to be dichotomous, there may be several winners and the result of a pairwise contest between two candidates depends on their approval scores. A candidate is then a Condorcet winner if he has the highest approval score.[12] Shapley ranking instead does not satisfy the Condorcet principle, as shown in Example 1 where candidates 2 and 3 are Condorcet winners but none of them is elected under Shapley ranking. Example 3 confirms that outside dichotomous preferences, approval and Shapley both fail to satisfy the Condorcet principle.

What happens to collective preferences when the preferences of a single elector change? This is the object of the Monotonicity axiom.

Monotonicity Consider a preference profile such that candidate *i* is *collectively* preferred to *j*. If a elector who prefers *j* to *i* changes his mind in favor of candidate *i*, candidate *i* must remain collectively preferred to candidate *j*.

Proposition 6 Under assumption A1 both approval ordering and Shapley ordering satisfy Monotonicity.

Proof. Assume that $i \succ j$ while $j \succ_k i$ for some $k \in M$. There are three possible cases:

[12] This is Theorem 3.1 in Brams and Fishburn (2007, p. 38).

(a) $j \in N_k$ and $i \notin N_k$,

(b) $i, j \in N_k$,

(c) $i, j \notin N_k$.

Assume that elector *k* changes his mind and now prefer *i* to *j*. We denote by N_k' his modified approval set. In case (a), we have three possible cases:

(a1) $N_k' = N_k \setminus j$,

(a2) $N_k' = N_k \cup i$,

(a3) $N_k' = (N_k \cup i) \setminus j$.

In case (b), there are two possible cases:

(b1) $N_k' = N_k \setminus j$,

(b2) $N_k' = N_k$.

In case (c), there are also two possible cases:

(c1) $N_k' = N_k \cup i$,

(c2) $N_k' = N_k$.

Consider first approval voting. Initially, we have $AR_i > AR_j$. In cases (b2) and (c2), AR_i and AR_j remain unchanged. In cases (a1) and (b1), AR_i is unaffected while AR_j decreases by 1. In cases (a2) and (c1), AR_j is unaffected while AR_i increases by 1. In case (a3), AR_i increases by 1 and AR_j decreases by 1. Hence, $AR_i' > AR_j'$. Consider now Shapley ranking. Initially, we have $SR_i > SR_j$. In cases (b2) and (c2), SR_i and SR_j remain unchanged. In cases (a1), SR_i is unaffected while SR_j decreases by $1/n_k$. In case (a2), SR_j decreases by $1/n_k(1+n_k)$ while SR_i increases by $1/(1+n_k)$. In case (b1), SR_j decreases by $1/n_k$ while SR_i increases by $1/n_k(n_k-1)$. In case (c1), SR_j is unaffected while SR_i increases by $1/(1+n_k)$. In case (a3), SR_i increases by $1/n_k$ and SR_j decreases by $1/n_k$. Hence, $SR_i' > SR_j'$. ♦

5. Concluding remarks

Approval voting has its advantages and drawbacks like any other preference aggregation method, although most of its advantages cannot be formalized. The same conclusion applies to equal-and-even cumulative voting. However, equal-and-even cumulative voting may be preferable to approval voting because under the former, electors have an incentive to limit the number of candidates they decide to retain. Furthermore, if an elector lists several candidates, it is likely that the candidates retained will be "close" to each other in terms of preferences, in which assuming that electors are indifferent between the candidates listed in their ballots becomes more plausible. It would be interesting to conduct experiments within approval balloting in order to evaluate the effect of fractional votes.

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