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CONVEX HULL RESULTS FOR GENERALIZATIONS OF THE CONSTANT CAPACITY SINGLE NODE FLOW SET

Laurence A. Wolsey[1] and Hande Yaman[2]

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Abstract

For single node flow sets with fixed costs and constant capacities on the inflow and outflow arcs, a family of constant capacity flow covers are known to provide the convex hull in different special cases and are conjectured to provide it in the general case. Here we study more general mixed integer sets for which such single node flow cover inequalities suffice to give the convex hull. In particular we consider the case of a path in which each node has one (or several) incoming and outgoing arcs with constant capacities and fixed costs. This can be seen as a lot-sizing set with production and sales decisions driven by costs and prices and by the lower and upper bounds on stocks instead of being driven by demands as in the standard lot-sizing model. The approach we take is classical: we characterize the extreme points, derive tight extended formulations and project out the additional variables. Specifically we show that Fourier-Motzkin elimination, though far from elegant, can be used to carry out the non-trivial projections. The validity of the conjecture for the single node flow set follows from our results.

Keywords: single node flow set, flow cover inequalities, convex hull, extended formulation, Fourier-Motzkin elimination
Mathematics Subject Classification: 90C11, 90C26

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[1]CORE, Université Catholique de Louvain, Voie du Roman Pays 34, B-1348 Louvain-la-Neuve, Belgium.
email: laurence.wolsey@uclouvain.be
[2]ORSTAT, Faculty of Economics and Business, KU Leuven, 3000 Leuven, Belgium.
email: hande.yaman@kuleuven.be

1 Introduction

Given two disjoint sets N^+ and N^- and two integers b_1 and c, the constant capacity single node flow set is the set of solutions to the system

$$\begin{aligned}&x(N^+) - x(N^-) \le b_1,\\ &0 \le x_j \le cz_j \quad j \in N,\\ &z_j \in \{0,1\} \quad j \in N,\end{aligned}$$

where $N = N^+ \cup N^-$ and $x(N') = \sum_{j \in N'} x_j$ for $N' \subseteq N$. The following conjecture/claim is well-known and cited in [2]: When b_1 is not a multiple of c, the convex hull of the single node flow set is obtained by adding the flow cover inequalities

$$x(T) - (c-\lambda)z(T) \le \left\lfloor \frac{b_1}{c} \right\rfloor \lambda + x(N^- \setminus V) + \lambda z(V),$$

where $T \subseteq N^+$, $V \subseteq N^-$ and $\lambda = \left\lceil \frac{b_1}{c} \right\rceil c - b_1$. A proof for the case when $N^- = \emptyset$ is given in [6] and for the case when the z variables are nonnegative integers in [1].

We consider the following generalization. Let n be a positive integer and sets N_t^+ and N_t^- for $t = 1, \dots, n$ be such that $N_1^+ \subseteq \dots \subseteq N_n^+$ and $N_1^- \subseteq \dots \subseteq N_n^-$. We define $N_t = N_t^+ \cup N_t^-$ for $t = 1, \dots, n$, $N^+ = N_n^+$, $N^- = N_n^-$ and $N = N^+ \cup N^-$. To simplify the notation, for two integers n_1 and $n_2 \ge n_1$, we let $[n_1, n_2] = \{n_1, \dots, n_2\}$. Let $a_1, \dots, a_n$ and $b_1, \dots, b_n$ be integers so that $a_t \le b_t$ for all $t \in [1, n]$.

We define set X to be the set of solutions to the system

$$\begin{aligned}&a_t \le x(N_t^+) - x(N_t^-) \le b_t \quad t \in [1,n],\\ &0 \le x_j \le cz_j \quad j \in N,\\ &z_j \in \{0,1\} \quad j \in N.\end{aligned}$$

The constant capacity single node flow set is the special case with $n = 1$ and $a_1 = -\infty$. One case in which such sets arise is the constant capacity fixed charge network flow problem. A digraph $G = (V, A)$ is given with node-arc incidence matrix D, lower and upper bounds a_j, b_j on the net flow entering at node $j \in V$ and constraints $a \le Dx \le b$, $0 \le x \le cz, z \in \{0,1\}^{|A|}$ where x_{ij} is the flow on arc (i,j) and $z_{ij} = 1$ if capacity is installed on arc (i,j). Taking a cut-set $(V', V \setminus V')$, one obtains the aggregate constraint

$$a(V') \le x(V', V \setminus V') - x(V \setminus V', V') \le b(V')$$

that, along with the variable upper bound and integrality constraints produces a single-node flow set. It can also be seen as a capacitated version of the warehouse problem treated in [8].

We provide the convex hull descriptions of

1. $X_{\le}^{b}$ which is the special case of X where $a_t = -\infty$ and $b_t \mod c = b$ for all $t \in [1, n]$,
2. $X_{\ge}^{a}$ which is the special case of X where $a_t \mod c = a$ and $b_t = \infty$ for all $t \in [1, n]$,
3. X^{a_1, b_1} which is the special case of X where $n = 1$, $a_1 \le 0 \le b_1$ and $b_1 - a_1 \ge c$.

In these cases, it is sufficient to add the flow cover inequalities for some constant capacity single node flow set relaxations to obtain the convex hull. Unfortunately, this is not true for the general case. Consider the convex hull of solutions to

$$\begin{aligned} &-4 \leq x_1 - x_4 \leq 7, \\ &-4 \leq x_1 + x_2 + x_3 - x_4 - x_5 - x_6 \leq 7, \\ &0 \leq x_j \leq 10 z_j, z_j \in \{0,1\} \quad j \in [1,6]. \end{aligned}$$

The inequality

$$9x_1 + 7x_2 + 9x_3 - 6x_5 - 6x_6 - 63z_1 - 7z_2 - 21z_3 - 27z_4 - 9z_5 - 9z_6 \leq 42$$

is facet-defining for this polytope and it is not a flow cover inequality.

The second set we are interested in is set Y, which is the set of solutions to

$$a_t \leq s_t = x_{1t} - y_{1t} \leq b_t \quad t \in [1,n], \tag{1}$$

$$y_t \geq 0 \quad t \in [1,n], \tag{2}$$

$$0 \leq x_t \leq c z_t \quad t \in [1,n], \tag{3}$$

$$z_t \in \{0,1\} \quad t \in [1,n],$$

where for a vector $\alpha \in \mathbb{R}^n$ and $t \in [1,n]$, $\alpha_{1t} = \sum_{u=1}^{t} \alpha_u$.

Set Y can be viewed as a lot-sizing model that is not demand-driven. The production x_t in period t is limited by the capacity constraint, the stock s_t at the end of period t is constrained by the lower and upper bounds a_t, b_t respectively, z_t is one if there is a set-up in t and y_t is the amount sold in period t. Valid inequalities and formulations for standard lot-sizing with stock upper bounds are given, among others, in Atamturk and Kucukyavuz [3] and Wolsey [7].

We provide the following results

1. a description of the convex hull of $Y^{0,b}$ which is the the special case of set Y when $a_t = 0$ and $b_t \mod c = b$ for all $t \in [1,n]$,
2. a tight extended formulation for the set Z which is the set Y in the discrete case when $x_t = c z_t$ for all $t \in [1,n]$,
3. a description of the convex hull of $Z^{a,b}$ which is the special case of Z when $a_t = a$ and $b_t = b$ for all $t \in [1,n]$ with $a < 0 < b$.

We also believe that the following more general result holds:
Conjecture: If $a_t = a < 0 < b_t = b$ for all $t \in [1,n]$, $\text{conv}(Y)$ is described by the constraints (1)-(3), $z_t \leq 1$ for $t \in [1,n]$ plus inequalities

$$s_t - s_j \leq \lambda_{b-a} \left\lfloor \frac{b-a}{c} \right\rfloor + x(T) + (c - \lambda_{b-a}) z([j+1,t] \setminus T)$$

for all $j \in [1, n-1]$, $t \in [j+1, n]$ and $T \subseteq [j+1, t]$ where $\lambda_{b-a} = \left\lceil \frac{b-a}{c} \right\rceil c - (b-a)$, and

$$s_t \leq \lambda_b \left\lfloor \frac{b}{c} \right\rfloor + x(T) + (c - \lambda_b) z([1,t] \setminus T)$$

for all $t \in [1,n]$ and $T \subseteq [1,t]$ where $\lambda_b = \left\lceil \frac{b}{c} \right\rceil c - b$.

In [8] we studied the convex hulls associated with various simple warehouse problems in which c is very large. There we observed that the nontrivial facet defining inequalities are flow cover inequalities. To find the description of the convex hulls, we presented tight extended formulations based on properties of the extreme points and then projected out the additional variables using Fourier-Motzkin elimination. We use the same approach here.

2 Convex hull results for set X

We are first interested in the convex hull of the mixed integer set $X_{\leq}$ defined as

$$x(N_t^+) - x(N_t^-) \leq b_t \quad t \in [1,n], \tag{4}$$
$$0 \leq x_j \leq cz_j \quad j \in N, \tag{5}$$
$$z_j \in \{0,1\} \quad j \in N. \tag{6}$$

In the sequel, we prove that when $b_t \mod c = b$ for $t \in [1,n]$ and $b > 0$, $\text{conv}(X_{\leq}^b)$ is described by the constraints (4), (5), $z_j \leq 1$ for $j \in N$ and inequalities

$$x(T) - (c-\lambda)z(T) \leq \left\lfloor \frac{b_t}{c} \right\rfloor \lambda + x(N_t^- \setminus V) + \lambda z(V) \tag{7}$$

for $\emptyset \subset T \subseteq N_t^+$, $V \subseteq N_t^-$ and $t \in [1,n]$ where $\lambda = c - b$. If $b = 0$, then $\text{conv}(X_{\leq}^b)$ is described by (4), (5) and $z_j \leq 1$ for $j \in N$.

2.1 Extreme points and extended formulation

We present a property of the extreme points and then use this property to derive an extended formulation.

Proposition 1 *At an extreme point of* $\text{conv}(X_{\leq}^b)$*, there is at most one* $j \in N$ *with* $0 < x_j < c$*. If* $j \in N^+$*, then* $x_j = c - \lambda$ *and if* $j \in N^-$*, then* $x_j = \lambda$*.*

Proof. Let (x,z) be an extreme point of $\text{conv}(X_{\leq}^b)$ such that there exist $t_1 \in [1,n]$ and $t_2 \in [t_1,n]$ with $j_1 \in N_{t_1}$, $j_2 \in N_{t_2}$, $j_1 \neq j_2$, $0 < x_{j_1} < c$ and $0 < x_{j_2} < c$. Consider the largest such t_1 and t_2. Let ϵ be a very small positive number. Define e_j to be a vector of size $|N|$ with 1 and -1 in position j if $j \in N^+$ and $j \in N^-$, respectively and 0 everywhere else.

If $t_1 = t_2$ or if no constraint (4) is tight at (x,z) for $t \in [t_1, t_2 - 1]$ then $(x,z) = 1/2(x^1,z) + 1/2(x^2,z)$ with $x^1 = x + \epsilon e_{j_1} - \epsilon e_{j_2}$ and $x^2 = x - \epsilon e_{j_1} + \epsilon e_{j_2}$. Moreover (x^1,z) and (x^2,z) are both in $X_{\leq}^b$. Hence (x,z) is not an extreme point. Now suppose that $t_1 < t_2$ and that there exists $t' \in [t_1, t_2 - 1]$ for which constraint (4) is tight at (x,z). Then constraint (4) is not tight at (x,z) for any $t \geq t_2$ since $0 < x_{j_2} < c$ and variables $x_j \in \{0,c\}$ for all $j \in N \setminus (N_{t'} \cup \{j_2\})$. In this case $(x,z) = 1/2(x^1,z) + 1/2(x^2,z)$ with $x^1 = x + \epsilon e_{j_2}$ and $x^2 = x - \epsilon e_{j_2}$ and (x^1,z) and (x^2,z) are both in $X_{\leq}^b$, proving that (x,z) is not an extreme point. Hence, at an extreme point of $\text{conv}(X_{\leq}^b)$, there can be at most one $j \in N$ with $0 < x_j < c$.

Now let (x,z) be an extreme point of $\text{conv}(X_{\leq}^b)$ such that there exists one $j \in N$ with $0 < x_j < c$ and $j \in N_{t'}$. Then at least one constraint (4) for some $t \geq t'$ is tight at (x,z). For this constraint to be tight $x_j = c - \lambda$ if $j \in N^+$ and $x_j = \lambda$ if $j \in N^-$. $\square$

For $j \in N^+$, we define variable z_j^1 to be 1 if $x_j = c$ and 0 otherwise and variable z_j^2 to be 1 if $x_j = c - \lambda$ and 0 otherwise. Similarly, for $j \in N^-$, we define variable z_j^1 to be 1 if $x_j = c$ and 0 otherwise and variable z_j^2 to be 1 if $x_j = \lambda$ and 0 otherwise. We provide an extended formulation using these additional variables.

Proposition 2 *The following is an extended formulation for set* $X^b_{\leq}$*:*

$$z^1(N_t^+) - z^1(N_t^-) - z^2(N_t^-) \leq \left\lfloor \frac{b_t}{c} \right\rfloor \quad t \in [1, n], \tag{8}$$
$$z^2(N^+) + z^2(N^-) \leq 1, \tag{9}$$
$$z_j^1 + z_j^2 - z_j \leq 0 \quad j \in N, \tag{10}$$
$$z_j^1, z_j^2 \geq 0 \quad j \in N, \tag{11}$$
$$z_j \leq 1 \quad j \in N, \tag{12}$$
$$z_j^1, z_j^2, z_j \in \{0, 1\} \quad j \in N, \tag{13}$$
$$x_j = cz_j^1 + (c - \lambda)z_j^2 \quad j \in N^+, \tag{14}$$
$$x_j = cz_j^1 + \lambda z_j^2 \quad j \in N^-. \tag{15}$$

Proof. It is easy to see that for any extreme point (x, z) of $\text{conv}(X^b_{\leq})$, there exist z^1, z^2 such that (x, z, z^1, z^2) satisfies the system (9)-(15). (x, z, z^1, z^2) also satisfies (8) since this inequality is obtained by rewriting constraint (4) using the additional variables, dividing it by c and applying rounding. Now for any (x, z, z^1, z^2) that satisfies the system (8)-(15), (x, z) clearly satisfies (5)-(6). For $t \in [1, n]$,

$$\begin{aligned} x(N_t^+) - x(N_t^-) &= cz^1(N_t^+) + (c - \lambda)z^2(N_t^+) - cz^1(N_t^-) - \lambda z^2(N_t^-) \\ &= c(z^1(N_t^+) - z^1(N_t^-) - z^2(N_t^-)) + (c - \lambda)(z^2(N_t^+) + z^2(N_t^-)) \\ &\leq c\left\lfloor \frac{b_t}{c} \right\rfloor + c - \lambda = b_t. \end{aligned}$$

So (x, z) also satisfies (4) and is in $X^b_{\leq}$. This proves that (8)-(15) is an extended formulation for set $X^b_{\leq}$. □

Next we prove that the extended formulation is tight. For this purpose, we first give a very simple lemma.

Lemma 1 *Let*

$$\Delta = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}.$$

If $\delta^1, \delta^2 \in \Delta$*, then either* $\delta^1 + \delta^2 \in \Delta$ *or* $\delta^1 - \delta^2 \in \Delta$*.*

Proof. It suffices to check case by case. □

Proposition 3 *At the extreme points of the polytope defined by* (8)-(12)*,* z_j^1*,* z_j^2*,* z_j *are 0-1 for all* $j \in N$*.*

Proof. To prove this, we show that the matrix associated with constraints (8)-(10) is totally unimodular (TU).

Given an expression $\sum_{j \in C^+} w_j - \sum_{j \in C^-} w_j$ that is part of a 0,+1,-1 constraint and a partition (J^+, J^-) of a subset J of the variables where we associate weights +1 and -1 to the members of J^+ and J^- respectively, the *weight* associated to the expression is $|J^+ \cap C^+| + |J^- \cap C^-| - |J^+ \cap C^-| - |J^- \cap C^+|$.

We use the characterization of Ghouila-Houri [5]: Given a 0,+1,-1 matrix, the matrix is totally unimodular if, for any subset J of the columns, there exists a partition (J^+, J^-) of J such that the weight associated to each row is 0, +1 or -1.

We observe that each variable z_j only occurs once with other variables, and can be ignored. Thus we only need to consider the z_j^1, z_j^2 variables. In addition if both z_j^1 and z_j^2 lie in J, the two variables cannot be assigned the same weight due to constraints (10).

Given a subset J of the variables, we order them selecting first from N_1, then $N_2 \setminus N_1$, etc. Let $\chi_j(t)$ be the weight assigned to the left hand side of constraint (8) for t by taking just the variables $1, \ldots, j$ that are in J. If $j \in N_k$, we observe that $\chi_j(t) = \chi_j(k)$ for $t \geq k$ and $\chi_j(t) = \chi_{j'}(t)$ for $j' \geq j$ and $t < k$. In addition, if $j+1 \in N_\ell$, then $\chi_{j+1}(t) = \chi_j(t)$ for $t < \ell$. Similarly let η_j be the weight assigned to the left hand side of constraint (9) by taking the variables with indices $1, \ldots, j$ that are in J.

As induction hypothesis, suppose that there exists an assignment of the pairs of variables for $1, \ldots, j$ with $j \in N_k$ such that $\chi_j(t) \in \{0, +1, -1\}$ for $t < k$ and $(\chi_j(k), \eta_j) \in \Delta$. Consider the pair of variables (z_{j+1}^1, z_{j+1}^2) with $j+1 \in N_\ell$. Since $\chi_{j+1}(t) = \chi_j(t)$ for $t < \ell$, $\chi_j(t) = \chi_j(k)$ for $t \geq k$ and $\chi_j(t) \in \{0, +1, -1\}$ for $t \leq k$, we have $\chi_{j+1}(t) \in \{0, +1, -1\}$ for $t < \ell$. Now to show that the induction hypothesis holds for $j+1$, it is sufficient to show that we can assign weights to z_{j+1}^1 and z_{j+1}^2, whichever are in J, such that $(\chi_{j+1}(\ell), \eta_{j+1}) \in \Delta$. If z_{j+1}^1 is in J, we assign it weight +1 and if z_{j+1}^2 is in J, we assign it weight -1. It is easily checked that $(\chi_{j+1}(\ell) - \chi_j(\ell), \eta_{j+1} - \eta_j) \in \Delta$. Since $\chi_j(\ell) = \chi_j(k)$ and $(\chi_j(k), \eta_j) \in \Delta$, $(\chi_{j+1}(\ell), \eta_{j+1})$ is the sum of two vectors in Δ and thus by the lemma their sum or difference is also in Δ. So by possibly switching the assignment of the variables (z_{j+1}^1, z_{j+1}^2) lying in J, we have extended the assignment to $j+1$. Finally observe that when $j = 0$, $(0,0) \in \Delta$, so the result holds for $j = 0$. The claim follows by induction. $\square$

2.2 Projection

Next we project out the additional variables to obtain a linear description in the original space.

Theorem 1 $\mathrm{conv}(X_\leq^b)$ *is described by the constraints* (4), (5), $z_j \leq 1$ *for* $j \in N$ *and inequalities* (7) *for* $T \subseteq N_t^+$, $V \subseteq N_t^-$ *and* $t \in [1, n]$.

Proof. Substituting $(c - \lambda) z_j^2 = x_j - c z_j^1$ for $j \in N^+$ and $\lambda z_j^2 = x_j - c z_j^1$ for $j \in N^-$ in (8)-(12) and setting $\alpha_j = z_j^1$ for $j \in N^+$ and $\beta_j = z_j^1$ for $j \in N^-$, we obtain

$$\lambda \alpha(N_t^+) + (c - \lambda)\beta(N_t^-) \leq \left\lfloor \frac{b_t}{c} \right\rfloor \lambda + x(N_t^-) \quad t \in [1, n], \tag{16}$$

$$\lambda(x(N^+) - c\alpha(N^+)) + (c - \lambda)(x(N^-) - c\beta(N^-))) \leq \lambda(c - \lambda), \tag{17}$$

$$\lambda \alpha_j \geq x_j - (c - \lambda) z_j \quad j \in N^+, \tag{18}$$

$$(c - \lambda)\beta_j \geq x_j - \lambda z_j \quad j \in N^-, \tag{19}$$

$$x_j \geq c\alpha_j \quad j \in N^+, \tag{20}$$

$$x_j \geq c\beta_j \quad j \in N^-, \tag{21}$$

$$\alpha_j \geq 0 \quad j \in N^+ \tag{22}$$

$$\beta_j \geq 0 \quad j \in N^-, \tag{23}$$

$$z_j \leq 1 \quad j \in N, \tag{24}$$

where (16) and (17) come from (8) and (9), (18) and (19) come from (10), and (20) and (21) come from the non-negativity of z_j^2.

We prove the theorem by projecting out the variables α and β from the system (16)-(24). We first show that after projecting out the variables α_j and β_j for $j \in N \setminus N_k$, the resulting polytope is given by

$$\lambda\alpha(N_t^+) + (c-\lambda)\beta(N_t^-) \le \left\lfloor \frac{b_t}{c} \right\rfloor \lambda + x(N_t^-) \quad t \in [1,k], \tag{25}$$

$$\lambda\alpha(N_k^+) + (c-\lambda)\beta(N_k^-) + \sum_{j\in N_t^+\setminus N_k^+} (x_j - (c-\lambda)z_j)^+ + \sum_{j\in N_t^-\setminus N_k^-} (x_j - \lambda z_j)^+ \le \left\lfloor \frac{b_t}{c} \right\rfloor \lambda + x(N_t^-) \quad t \in [k+1,n], \tag{26}$$

$$\lambda x(N_k^+) + (c-\lambda)x(N_k^-) - \lambda c\alpha(N_k^+) - (c-\lambda)c\beta(N_k^-) \le \lambda(c-\lambda), \tag{27}$$

$$x_j \ge c\alpha_j \quad j \in N_k^+, \tag{28}$$

$$x_j \ge c\beta_j \quad j \in N_k^-, \tag{29}$$

$$\lambda\alpha_j \ge (x_j - (c-\lambda)z_j)^+ \quad j \in N_k^+, \tag{30}$$

$$(c-\lambda)\beta_j \ge (x_j - \lambda z_j)^+ \quad j \in N_k^-, \tag{31}$$

$$x(N_t^+) - x(N_t^-) \le b_t \quad t \in [k+1,n], \tag{32}$$

$$x_j \le cz_j \quad j \in N \setminus N_k, \tag{33}$$

$$x_j \ge 0 \quad j \in N \setminus N_k, \tag{34}$$

$$z_j \le 1 \quad j \in N. \tag{35}$$

We prove this result by induction. When $k = n$, we obtain the system (16)-(24). Now we first project out α_u for some $u \in N_k^+ \setminus N_{k-1}^+$ using Fourier-Motzkin elimination.

Combining inequality (28) with (30) give $x_u \ge 0$ and $x_u \le cz_u$. Combining inequality (28) with (27) gives

$$\lambda x(N_k^+ \setminus \{u\}) + (c-\lambda)x(N_k^-) - \lambda c\alpha(N_k^+ \setminus \{u\}) - (c-\lambda)c\beta(N_k^-) \le \lambda(c-\lambda).$$

We treat (25) for $t = k$ and (26) for $t \in [k+1,n]$ together. Combining them with (30) gives

$$\lambda\alpha(N_k^+ \setminus \{u\}) + (c-\lambda)\beta(N_k^-) + \sum_{j\in N_t^+\setminus N_k^+\cup\{u\}} (x_j - (c-\lambda)z_j)^+ + \sum_{j\in N_t^-\setminus N_k^-} (x_j - \lambda z_j)^+ \le \left\lfloor \frac{b_t}{c} \right\rfloor \lambda + x(N_t^-) \quad t \in [k,n].$$

Combining (25) for $t = k$ and (26) for $t \in [k+1,n]$ with (27) gives

$$\lambda x(N_k^+) + (c-\lambda)x(N_k^-) + c\sum_{j\in N_t^+\setminus N_k^+} (x_j - (c-\lambda)z_j)^+ + c\sum_{j\in N_t^-\setminus N_k^-} (x_j - \lambda z_j)^+ \le \lambda b_t + cx(N_t^-) \quad t \in [k,n]. \tag{36}$$

For $t = k$, inequality (36) gives $x(N_k^+) - x(N_k^-) \le b_k$. Now we prove that (36) is dominated when $t > k$. Inequality (36) is equivalent to the family of inequalities:

$$\lambda x(N_k^+) + (c-\lambda)x(N_k^-) + c\sum_{j\in T}(x_j - (c-\lambda)z_j) + c\sum_{j\in V}(x_j - \lambda z_j) \le \lambda b_t + cx(N_t^-) \tag{37}$$

for all $T \subseteq N_t^+ \setminus N_k^+$ and $V \subseteq N_t^- \setminus N_k^-$. For given T and V, this inequality can be obtained by taking $x(N_t^+) - x(N_t^-) \le b_t$ with weight λ, $-x_j \le 0$ for $j \in N_t^+ \setminus (N_k^+ \cup T)$ with weight λ, $x_j - cz_j \le 0$ for $j \in T$ with weight $c - \lambda$, $-x_j \le 0$ for $j \in N_t^- \setminus (N_k^- \cup V)$ with weight $c - \lambda$ and $x_j - cz_j \le 0$ for $j \in V$ with weight λ.

The elimination for all α_u's with $u \in N_k^+ \setminus N_{k-1}^+$ is the same. After eliminating these variables, we obtain the polytope given by

$$\lambda\alpha(N_t^+) + (c-\lambda)\beta(N_t^-) \le \left\lfloor \frac{b_t}{c} \right\rfloor \lambda + x(N_t^-) \quad t \in [1, k-1], \tag{38}$$

$$\lambda\alpha(N_{k-1}^+) + (c-\lambda)\beta(N_k^-) + \sum_{j \in N_t^+ \setminus N_{k-1}^+} (x_j - (c-\lambda)z_j)^+ + \sum_{j \in N_t^- \setminus N_k^-} (x_j - \lambda z_j)^+ \le \left\lfloor \frac{b_t}{c} \right\rfloor \lambda + x(N_t^-) \quad t \in [k, n], \tag{39}$$

$$\lambda x(N_{k-1}^+) + (c-\lambda)x(N_k^-) - \lambda c\alpha(N_{k-1}^+) - (c-\lambda)c\beta(N_k^-) \le \lambda(c-\lambda), \tag{40}$$

$$x_j \ge c\alpha_j \quad j \in N_{k-1}^+, \tag{41}$$

$$x_j \ge c\beta_j \quad j \in N_k^-, \tag{42}$$

$$\lambda\alpha_j \ge (x_j - (c-\lambda)z_j)^+ \quad j \in N_{k-1}^+, \tag{43}$$

$$(c-\lambda)\beta_j \ge (x_j - \lambda z_j)^+ \quad j \in N_k^-, \tag{44}$$

$$x(N_t^+) - x(N_t^-) \le b_t \quad t \in [k, n], \tag{45}$$

$$x_j \le cz_j \quad j \in (N \setminus N_k) \cup N_k^+, \tag{46}$$

$$x_j \ge 0 \quad j \in (N \setminus N_k) \cup N_k^+, \tag{47}$$

$$z_j \le 1 \quad j \in N. \tag{48}$$

The next step is to eliminate β_u with $u \in N_k^- \setminus N_{k-1}^-$. Combining (44) and (42) gives $x_u \ge 0$ and $x_u \le cz_u$. Combining (40) and (42) gives

$$\lambda x(N_{k-1}^+) + (c-\lambda)x(N_k^- \setminus \{u\}) - \lambda c\alpha(N_{k-1}^+) - (c-\lambda)c\beta(N_k^- \setminus \{u\}) \le \lambda(c-\lambda).$$

Combining (44) with (39) for $t \in [k, n]$ gives

$$\lambda\alpha(N_{k-1}^+) + (c-\lambda)\beta(N_k^- \setminus \{u\}) + \textstyle\sum_{j \in N_t^+ \setminus N_{k-1}^+} (x_j - (c-\lambda)z_j)^+ + \sum_{j \in N_t^- \setminus N_k^- \cup \{u\}} (x_j - \lambda z_j)^+ \le \left\lfloor \frac{b_t}{c} \right\rfloor \lambda + x(N_t^-) \quad t \in [k, n].$$

Finally, we combine (40) with (39) for $t \in [k, n]$ to obtain

$$\lambda x(N_{k-1}^+) + (c-\lambda)x(N_k^-) + c\sum_{j \in N_t^+ \setminus N_{k-1}^+} (x_j - (c-\lambda)z_j)^+ + c\sum_{j \in N_t^- \setminus N_k^-} (x_j - \lambda z_j)^+ \le \lambda b_t + cx(N_t^-) \quad t \in [k, n].$$

It can be proved as above that these inequalities are redundant. All variables β_u with $u \in N_k^- \setminus N_{k-1}^-$ are projected out in the same way.

We obtain the result of the theorem for $k = 0$ with $N_0^+ = N_0^- = N_0 = \emptyset$. In this case, inequalities (26), which are the only nontrivial inequalities, become

$$\sum_{j \in N_t^+} (x_j - (c-\lambda)z_j)^+ + \sum_{j \in N_t^-} (x_j - \lambda z_j)^+ \le \left\lfloor \frac{b_t}{c} \right\rfloor \lambda + x(N_t^-) \quad t \in [1, n],$$

which are equivalent to the flow cover inequalities (7). □

With $n = 1$, observe that the single node flow set conjecture from the first paragraph is now proven.

2.3 Extensions

As we did not assume that b_t's are positive, the following theorem can be derived from the above results:

Theorem 2 *Let $X_{\geq}$ be the set of solutions to the system*

$$\begin{aligned} & x(N_t^+) - x(N_t^-) \geq a_t \quad t \in [1, n], \\ & 0 \leq x_j \leq c z_j \quad j \in N, \\ & z_j \in \{0, 1\} \quad j \in N. \end{aligned}$$

If $a_t \mod c = a$ for $t = 1, \ldots, n$, then $\text{conv}(X_{\geq}^a)$ is described by the original constraints and the inequalities

$$x(N_t^+ \setminus T) + a z(T) \geq a \left\lceil \frac{a_t}{c} \right\rceil + x(V) - (c - a) z(V) \tag{49}$$

for all $t \in [1, n]$, $T \subseteq N_t^+$ and $V \subseteq N_t^-$.

Set X is the intersection of sets $X_{\geq}$ and $X_{\leq}$. As shown by the example in the Introduction, $\text{conv}(X) \neq \text{conv}(X_{\geq}) \cap \text{conv}(X_{\leq})$ in general. However, this holds in some special cases, in particular when $n = 1$.

Theorem 3 *If $n = 1$, $a = a_1 \leq 0 \leq b = b_1$ and $b - a \geq c$, then $\text{conv}(X^{a_1,b_1}) = \text{conv}(X_{\geq}^{a_1}) \cap \text{conv}(X_{\leq}^{b_1})$.*

Proof. Suppose that we are minimizing $\sum_{j \in N}(f_j z_j + g_j x_j)$ over sets X^{a_1,b_1}, $X_{\leq}^{b_1}$ and $X_{\geq}^{a_1}$ with (f, g) not equal to zero. Let $X^{a_1,b_1}(f, g)$, $X_{\leq}^{b_1}(f, g)$ and $X_{\geq}^{a_1}(f, g)$ be the respective sets of all optimal solutions. We would like to show that $X^{a_1,b_1}(f, g) \subseteq X_{\geq}^{a_1}(f, g)$ or $X^{a_1,b_1}(f, g) \subseteq X_{\leq}^{b_1}(f, g)$ and hence all optimal solutions lie on a face defined by one of the facet defining inequalities of $\text{conv}(X_{\geq}^{a_1})$ or $\text{conv}(X_{\leq}^{b_1})$.

Suppose to the contrary that $X^{a_1,b_1}(f, g) \not\subseteq X_{\geq}^{a_1}(f, g)$ and $X^{a_1,b_1}(f, g) \not\subseteq X_{\leq}^{b_1}(f, g)$.

There exists $(x', z') \in X^{a_1,b_1}(f, g) \backslash X_{\geq}^{a_1}(f, g)$. Since $(x', z') \notin X_{\geq}^{a_1}(f, g)$, $\min_{(x,z) \in X} \sum_{j \in N}(f_j z_j + g_j x_j) > \min_{(x,z) \in X_{\geq}^{a_1}} \sum_{j \in N}(f_j z_j + g_j x_j)$. Then $X^{a_1,b_1}(f, g) \cap X_{\geq}^{a_1}(f, g) = \emptyset$ and consequently $x(N^+) - x(N^-) > b$ for all $(x, z) \in X_{\geq}^{a!}(f, g)$. In this case, at extreme points of $\text{conv}(X_{\geq}^{a_1})$ that are in $X_{\geq}^{a_1}(f, g)$, $x_j \in \{0, c\}$ for all $j \in N$.

Let $N_{>}^+ = \{j \in N^+ : f_j + c g_j > 0\}$, $N_{<}^+ = \{j \in N^+ : f_j + c g_j < 0\}$ and $N_0^+ = \{j \in N^+ : f_j + c g_j = 0\}$. Similarly, $N_{>}^- = \{j \in N^- : f_j + c g_j > 0\}$, $N_{<}^- = \{j \in N^- : f_j + c g_j < 0\}$ and $N_0^- = \{j \in N^- : f_j + c g_j = 0\}$. If $(x, z) \in X_{\geq}^{a_1}(f, g)$ and it is an extreme point of $\text{conv}(X_{\geq}^{a_1})$, then we should have $x_j = c$ for all $j \in N_{<}^+ \cup N_{<}^-$ and $x_j = 0$ for all $j \in N_{>}^+ \cup N_{>}^-$ since if there exists $j \in N_{<}^+ \cup N_{<}^-$ with $x_j = 0$ (respectively, $j \in N_{>}^+ \cup N_{>}^-$ with $x_j = c$) then setting $z_j = 1$ and $x_j = c$ (respectively, setting $z_j = 0$ and $x_j = 0$) gives a feasible solution with a better objective function value. Hence $|N_{<}^+| - |N_{<}^-| \geq \lceil \frac{b}{c} \rceil$.

There exists also $(x', z') \in X^{a_1,b_1}(f, g) \backslash X_{\leq}^{b_1}(f, g)$. Following the same arguments, we get that $|N_{<}^+| - |N_{<}^-| \leq \lfloor \frac{a}{c} \rfloor$. We obtain $\lceil \frac{b}{c} \rceil \leq \lfloor \frac{a}{c} \rfloor$, which is a contradiction. So $X^{a_1,b_1}(f, g) \subseteq X_{\geq}^{a_1}(f, g)$ or $X^{a_1,b_1}(f, g) \subseteq X_{\leq}^{b_1}(f, g)$. □

3 Convex hull results for set $Y^{0,b}$

Constraints $s_t = x_{1t} - y_{1t}$ for $t \in [1,n]$ are equivalent to $s_t = s_{t-1} + x_t - y_t$ for $t \in [1,n]$ with $s_0 = 0$. So set Y is the feasible set of the system:

$$s_t = s_{t-1} + x_t - y_t \quad t \in [1,n], \tag{50}$$
$$s_0 = 0, \tag{51}$$
$$0 \le s_t \le b_t \quad t \in [1,n], \tag{52}$$
$$0 \le x_t \le cz_t \quad t \in [1,n], \tag{53}$$
$$y_t \ge 0 \quad t \in [1,n], \tag{54}$$
$$z_t \in \{0,1\} \quad t \in [1,n]. \tag{55}$$

As in the previous section, we define $\lambda = c - b$.

3.1 Extreme points and extended formulation

Proposition 4 *At an extreme point of* $\text{conv}(Y^{0,b})$, $x_t \in \{0, c-\lambda, c\}$ *and* $s_t \mod c \in \{0, c-\lambda\}$ *for* $t \in [1,n]$. *In addition, if* $x_t = c - \lambda$ *for some* $t \in [1,n]$ *then* $s_{t-1} \mod c = 0$ *and* $s_t \mod c = c - \lambda$.

Proof. The system can be modeled as a network flow on a directed graph where the set of nodes is $[0,n]$. For $t \in [1,n]$, we add an arc from node 0 to node t with flow x_t and an arc from node t to node 0 with flow y_t. We also add arcs $(t, t+1)$ with flow s_t for $t \in [1, n-1]$ and an arc from node n to node 0 with flow s_n. Let (x,y,s,z) be an extreme point of $\text{conv}(Y^{0,b})$. The set $[1, n+1]$ can be decomposed into intervals $[t_1, t_2]$ with $1 \le t_1 \le t_2 \le n+1$ such that s_{t_1-1} and s_{t_2} are at one of their bounds and $0 < s_t < b_t$ for $t \in [t_1, t_2 - 1]$ (s_0 and s_{n+1} are at their bounds by definition). If $t_2 = n+1$, then none of the variables x_t and y_t for $t \in [t_1, n]$ can be basic (otherwise basic arcs form a cycle). This implies that $x_t \in \{0, c\}$, $y_t = 0$ and consequently $s_t \mod c = s_{t_1-1} \mod c$ for all $t \in [t_1, n]$.

If $t_2 \le n$, then we may have at most one $l \in [t_1, t_2]$ with $0 < x_l < c$ or $y_l > 0$ but not both. If $s_{t_1-1} \mod c = s_{t_2} \mod c$, then $x_t \in \{0, c\}$ for all $t \in [t_1, t_2]$. Also $s_t \mod c = s_{t_1-1} \mod c$ for all $t \in [t_1, t_2]$. If $s_{t_1-1} \mod c \ne s_{t_2} \mod c$, then the possibilities are $s_{t_1-1} = 0, s_{t_2} = b_{t_2}$ and $s_{t_1-1} = b_{t_1-1}, s_{t_2} = 0$. In the first case, there exists an $l \in [t_1, t_2]$ with $x_l = c - \lambda$ or $y_l \mod c = \lambda$. In the second case, since the stock changes from b_{t_1-1} to zero, there exists an $l \in [t_1, t_2]$ with $y_l > 0$ and $x_t \in \{0, c\}$ in all $t \in [t_1, t_2]$. In both cases, $s_t \mod c = s_{t_1-1} \mod c$ for $t \in [t_1, l-1]$ and $s_t \mod c = s_{t_2} \mod c$ for $t \in [l, t_2]$.

In all cases, we have $s_t \mod c \in \{0, c-\lambda\}$ for $t \in [1,n]$. Also if $x_t = c - \lambda$ then $s_{t-1} \mod c = 0$ and $s_t \mod c = c - \lambda$. □

Next we introduce the following variables to obtain an extended formulation for set Y:
$\alpha_t^1 = 1$ if $x_t = c$ and 0 otherwise,
$\alpha_t^2 = 1$ if $x_t = c - \lambda$ and 0 otherwise,
$\sigma_t^1 = \lfloor \frac{s_t}{c} \rfloor$: the integer multiples of c of s_t,
$\sigma_t^2 = 1$ if $s_t \mod c = c - \lambda$ and 0 otherwise.
With these new variables, we have $x_t = c\alpha_t^1 + (c-\lambda)\alpha_t^2$ and $s_t = c\sigma_t^1 + (c-\lambda)\sigma_t^2$ for $t \in [1,n]$.

Proposition 5 *The following is an extended formulation for set $Y^{0,b}$:*

$$x_t = c\alpha_t^1 + (c-\lambda)\alpha_t^2 \quad t \in [1,n], \tag{56}$$
$$s_t = c\sigma_t^1 + (c-\lambda)\sigma_t^2 \quad t \in [1,n], \tag{57}$$
$$y_t = s_{t-1} + x_t - s_t \quad t \in [1,n], \tag{58}$$
$$s_0 = \sigma_0^1 = \sigma_0^2 = 0, \tag{59}$$
$$\sigma_{t-1}^1 + \alpha_t^1 - \sigma_t^1 \geq 0 \quad t \in [1,n], \tag{60}$$
$$\sigma_{t-1}^1 + \sigma_{t-1}^2 + \alpha_t^1 + \alpha_t^2 - \sigma_t^1 - \sigma_t^2 \geq 0 \quad t \in [1,n], \tag{61}$$
$$\alpha_t^1 + \alpha_t^2 - z_t \leq 0 \quad t \in [1,n], \tag{62}$$
$$\alpha_t^2 - \sigma_t^2 \leq 0 \quad t \in [1,n], \tag{63}$$
$$\sigma_t^1 \leq \left\lfloor \frac{b_t}{c} \right\rfloor \quad t \in [1,n], \tag{64}$$
$$\sigma_t^1 \geq 0 \quad t \in [1,n], \tag{65}$$
$$0 \leq \alpha_t^1, \alpha_t^2, \sigma_t^2, z_t \leq 1 \quad t \in [1,n], \tag{66}$$
$$\alpha_t^1, \alpha_t^2, \sigma_t^1, \sigma_t^2, z_t \ \textit{integer} \quad t \in [1,n]. \tag{67}$$

In addition the polytope defined by (59)-(66) *is integral.*

Proof. Let (x,s,y,z) be an extreme point of $conv(Y^{0,b})$. From Proposition 4, we have that there exist $\alpha^1, \alpha^2, \sigma^1$ and σ^2 such that $x_t = c\alpha_t^1 + (c-\lambda)\alpha_t^2$ and $s_t = c\sigma_t^1 + (c-\lambda)\sigma_t^2$ with $\sigma_t^1 \geq 0$ and integer and $\alpha_t^1, \alpha_t^2, \sigma_t^2 \in \{0,1\}$ for $t \in [1,n]$. The fact that if $x_t = c - \lambda$ then $s_t \bmod c = c - \lambda$ implies $\alpha_t^2 \leq \sigma_t^2$ for $t \in [1,n]$.

Let $t \in [1,n]$. If $\alpha_t^1 = 1$, then we have $s_{t-1} + c \geq s_t$ implying that $\sigma_{t-1}^1 + 1 \geq \sigma_t^1$. If $\alpha_t^1 = 0$ and $x_t = 0$, then $s_{t-1} + 0 \geq s_t$ implying that $\sigma_{t-1}^1 \geq \sigma_t^1$. If $\alpha_t^1 = 0$ and $x_t = c - \lambda$, then $s_{t-1} \bmod c = 0$. Therefore $s_{t-1} = c\sigma_{t-1}^1$, $s_t \leq c\sigma_{t-1}^1 + (c-\lambda)$ and thus $\sigma_t \leq \sigma_{t-1}$. Thus in every case $\sigma_{t-1}^1 + \alpha_t^1 \geq \sigma_t^1$.

The inequality $c\sigma_{t-1}^1 + (c-\lambda)\sigma_{t-1}^2 + c\alpha_t^1 + (c-\lambda)\alpha_t^2 - c\sigma_t^1 - (c-\lambda)\sigma_t^2 \geq 0$ can be rewritten as $(c-\lambda)(\sigma_{t-1}^1 + \sigma_{t-1}^2 + \alpha_t^1 + \alpha_t^2 - \sigma_t^1 - \sigma_t^2) + \lambda(\sigma_{t-1}^1 + \alpha_t^1 - \sigma_t^1) \geq 0$. We know that $\sigma_{t-1}^1 + \alpha_t^1 - \sigma_t^1 \geq 0$. If $\sigma_{t-1}^1 + \alpha_t^1 - \sigma_t^1 = 0$, the above inequality implies $\sigma_{t-1}^1 + \sigma_{t-1}^2 + \alpha_t^1 + \alpha_t^2 \geq \sigma_t^1 + \sigma_t^2$. On the other hand if $\sigma_{t-1}^1 + \alpha_t^1 - \sigma_t^1 \geq 1$, then as $\sigma_{t-1}^2 + \alpha_t^2 - \sigma_t^2 \geq -1$, we again have $\sigma_{t-1}^1 + \sigma_{t-1}^2 + \alpha_t^1 + \alpha_t^2 \geq \sigma_t^1 + \sigma_t^2$.

For $t \in [1,n]$, constraint $s_t \leq b_t$ is the same as $c\sigma_t^1 + (c-\lambda)\sigma_t^2 \leq b_t$ which implies $\sigma_t^1 \leq \left\lfloor \frac{b_t}{c} \right\rfloor$. Also constraint $x_t \leq cz_t$, which is the same as $c\alpha_t^1 + (c-\lambda)\alpha_t^2 \leq cz_t$, gives $\alpha_t^1 + \alpha_t^2 \leq z_t$.

Hence for each extreme point (x,s,y,z) of $conv(Y^{0,b})$, there exist $\alpha^1, \alpha^2, \sigma^1$ and σ^2 such that $(x,s,y,z,\alpha^1,\alpha^2,\sigma^1,\sigma^2)$ is feasible in the extended formulation.

Next, we show that for any $(x,s,y,z,\alpha^1,\alpha^2,\sigma^1,\sigma^2)$ that is feasible in the extended formulation, $(x,s,y,z) \in Y$. It is easy to see that (x,s,y,z) satisfies (50)-(53) and (**??**). To show that $y_t \geq 0$, observe that $y_t = (c-\lambda)(\sigma_{t-1}^1 + \sigma_{t-1}^2 + \alpha_t^1 + \alpha_t^2 - \sigma_t^1 - \sigma_t^2) + \lambda(\sigma_{t-1}^1 + \alpha_t^1 - \sigma_t^1)$. As both $\sigma_{t-1}^1 + \sigma_{t-1}^2 + \alpha_t^1 + \alpha_t^2 - \sigma_t^1 - \sigma_t^2$ and $\sigma_{t-1}^1 + \alpha_t^1 - \sigma_t^1$ are nonnegative, the result follows.

To prove that the polytope defined by (59)-(66) is integral, we show that the constraint matrix associated with (60)-(63) is totally unimodular. We again use the characterization of Ghoula-Houri. Let Δ^* be the same as the set Δ of Lemma 1, except that the last two vectors are replaced by $\begin{pmatrix} 1 & 1 \end{pmatrix}^T, \begin{pmatrix} -1 & -1 \end{pmatrix}^T$. Lemma 1 also holds for Δ^*.

First we observe that the variables z_t just occur once and can thus be ignored.

Consider a subset J of the remaining variables and let J_{k-1} be those variables in J with subscript at most $k-1$. The induction hypothesis is that there exists a partition (J_{k-1}^+, J_{k-1}^-) of J_{k-1} with associated weights $(+1,-1)$ such that

i) the weights associated to the constraints (60)-(63) for $t \leq k-1$ lie in $\{0,+1,-1\}$

ii) the weights δ^1 associated to the terms $(\sigma^1_{k-1}, \sigma^1_{k-1} + \sigma^2_{k-1})$ from (60)-(61) for $t = k$ are such that $\delta^1 \in \Delta^*$.

Now we show that we can associate weights to J_k such that the induction hypothesis holds for k. If present in $J_k \setminus J_{k-1}$, we assign the variables with σ^2_k and α^2_k to the same set so that the weight associated to (63) lies in $\{0, 1, -1\}$. Also α^1_k and α^2_k are assigned to different sets of the partition so that the weight associated to (62) lies in $\{0, 1, -1\}$. Now let δ^2 be the weight associated to the terms $(\alpha^1_k - \sigma^1_k, \alpha^1_k + \alpha^2_k - \sigma^1_k - \sigma^2_k)$ from (60)-(61) for $t = k$. First we observe that if less than two of the subscript k variables lie in J_k, then $\delta^2 \in \Delta^*$. Below we propose 11 possible assignments when two or more of the subscript k variables lie in J_k.

α^1_k	α^2_k	σ^1_k	σ^2_k	δ^2_1	δ^2_2
J^+	J^-	J^+	J^-	0	0
J^+	J^-	J^+	.	0	-1
J^-	J^+	.	J^+	-1	-1
J^+	.	J^+	J^-	0	+1
.	J^+	J^-	J^+	+1	+1
J^+	J^-	.	.	+1	0
J^+	.	J^+	.	0	0
J^+	.	.	J^+	+1	0
.	J^+	J^+	.	-1	0
.	J^+	.	J^+	0	0
.	.	J^+	J^-	-1	0

We have thus verified case by case that $\delta^2 \in \Delta^*$. Now δ^1 and δ^2 can be combined to give a vector in Δ^* showing that there is a partition of J_k in which the weight assigned to rows (60)-(61) for $t = k$ lies in $\{0, 1, -1\}$. Also none of the assignments assign σ^1_k and σ^2_k to the same set, and thus the weight δ^1 associated to $(\sigma^1_k, \sigma^1_k + \sigma^2_k)$ lies in Δ^*.

Finally the induction hypothesis holds for $k = 1$ taking $\sigma^1_0 = \sigma^2_0 = 0$ and thus the claim follows by induction. □

3.2 Projection

Theorem 4 $\text{conv}(Y^{0,b})$ *is described by constraints* (50)-(53), $z_t \leq 1$ *for* $t \in [1, n]$ *and inequalities*

$$s_t - s_j \leq \lambda \left\lfloor \frac{b_t}{c} \right\rfloor + \sum_{u=j+1}^{t} \min\{x_u, (c-\lambda)z_u\} \quad j \in [1, n], t \in [j+1, n],$$

$$s_t \leq \lambda \left\lfloor \frac{b_t}{c} \right\rfloor + \sum_{u=1}^{t} \min\{x_u, (c-\lambda)z_u\} \quad t \in [1, n].$$

Proof. We start with the extended formulation (56)-(66) and we project out the variables $\alpha^1_t, \alpha^2_t, \sigma^1_t, \sigma^2_t$ variables working down from $t = n, n-1, \ldots, 1$.

Consider the sets R^k

$$
\begin{aligned}
x_t &\leq cz_t \quad t \in [k+1, n], && (68)\\
x_t &\geq 0 \quad t \in [k+1, n], && (69)\\
s_t &\geq 0 \quad t \in [k+1, n], && (70)\\
s_t &\leq b_t \quad t \in [k+1, n], && (71)\\
s_t &\leq s_{t-1} + x_t \quad t \in [k+1, n], && (72)\\
s_t - s_j &\leq \lambda \left\lfloor \frac{b_t}{c} \right\rfloor + \sum_{u=j+1}^{t} \min\{x_u, (c-\lambda)z_u\} \quad j \in [k+1, n], t \in [j+1, n], && (73)\\
s_t &\leq \lambda \left\lfloor \frac{b_t}{c} \right\rfloor + \sum_{u=k+1}^{t} \min\{x_u, (c-\lambda)z_u\} + (c-\lambda)(\sigma_k^1 + \sigma_k^2) \quad t \in [k+1, n], && (74)
\end{aligned}
$$

and Q^k

$$x_t = c\alpha_t^1 + (c-\lambda)\alpha_t^2 \quad t \in [1, k], \tag{75}$$
$$s_t = c\sigma_t^1 + (c-\lambda)\sigma_t^2 \quad t \in [1, k], \tag{76}$$
$$s_0 = \sigma_0^1 = \sigma_0^2 = 0, \tag{77}$$
$$\sigma_{t-1}^1 + \alpha_t^1 - \sigma_t^1 \geq 0 \quad t \in [1, k], \tag{78}$$
$$\sigma_{t-1}^1 + \sigma_{t-1}^2 + \alpha_t^1 + \alpha_t^2 - \sigma_t^1 - \sigma_t^2 \geq 0 \quad t \in [1, k], \tag{79}$$
$$\alpha_t^1 + \alpha_t^2 - z_t \leq 0 \quad t \in [1, k], \tag{80}$$
$$\alpha_t^2 - \sigma_t^2 \leq 0 \quad t \in [1, k], \tag{81}$$
$$\sigma_t^1 \leq \left\lfloor \tfrac{b_t}{c} \right\rfloor \quad t \in [1, k], \tag{82}$$
$$\sigma_t^2 \leq 1 \quad t \in [1, k], \tag{83}$$
$$\alpha_t^2, \sigma_t^1 \geq 0 \quad t \in [1, k], \tag{84}$$
$$\alpha_t^1 \geq 0 \quad t \in [1, k], \tag{85}$$
$$z_t \leq 1 \quad t \in [1, n]. \tag{86}$$

Note that $Q^n \cap R^n$ is the original system and $Q^0 \cap R^0$ is the final system with variables y_t projected out.

We prove by induction that after eliminating the variables with subscripts $t = n, n-1, \ldots, k+1$, the resulting projection is $Q^k \cap R^k$. The proof is given in Appendix 1. □

4 Convex hull results for the discrete lot-sizing set Z

The set Z can be written in the form

$$s_{t-1} + cz_t \geq s_t \quad t \in [1, n], \tag{87}$$
$$s_0 = 0, \tag{88}$$
$$s_t \leq b_t \quad t \in [1, n], \tag{89}$$
$$s_t \geq a_t \quad t \in [1, n], \tag{90}$$
$$z_t \in \{0, 1\} \quad t \in [1, n]. \tag{91}$$

Note that the convex hull of Z is a face of the convex hull of Y. We assume that $a_t \leq 0 < b_t$, $b_t = \min\{b_t, b_{t-1} + c\}$ and $a_t = \max\{a_t, a_{t+1} - c\}$ for all $t \in [1, n]$ with $b_0 = 0$ and $a_{n+1} = -\infty$.

4.1 Extreme points and a tight extended formulation for Z

Let $F = \{\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n, 0\}$, where $\alpha_j = a_j \mod c$ and $\beta_j = b_j \mod c$ for all $j \in [1, n]$.

Observation 1 *In an extreme point of* $\text{conv}(Z)$*,* $s_t \mod c \in F$ *for all* $t \in [1, n]$*.*

Observation 2 *With the change of variable* $\zeta_t = -z_{1t}$ *and* $\sigma_t = s_t - cz_{1t} = s_t + c\zeta_t$*, we obtain the equivalent model:*

$$\sigma_t - \sigma_{t-1} \leq 0 \quad t \in [1, n], \tag{92}$$
$$\sigma_t - c\zeta_t \leq b_t \quad t \in [1, n], \tag{93}$$
$$\sigma_t - c\zeta_t \geq a_t \quad t \in [1, n], \tag{94}$$
$$\zeta_t - \zeta_{t-1} \leq 0 \quad t \in [1, n], \tag{95}$$
$$\zeta_t - \zeta_{t-1} \geq -1 \quad t \in [1, n], \tag{96}$$
$$\sigma_0 = \zeta_0 = 0, \tag{97}$$
$$\zeta_t \in \mathbb{Z} \quad t \in [1, n].$$

This is a dual network MIP, see [4]. From Observation 1, σ_t/c can only take at most the $K + 1$ distinct fractional values in F with $K \leq 2n$ and thus from Section 4 in [4] one obtains the tight polynomial size extended formulation of the convex hull that we now describe.

We order the elements of $F \setminus \{0\}$ in decreasing order, i.e., $f_0 = c > f_1 > \cdots f_K > f_{K+1} = 0$ Let ℓ_t and k_t be the positions of α_t and β_t, respectively, in this ordering.

The extended formulation is

$$\begin{aligned}
\sigma_t &= \sum_{j=0}^{K} (f_j - f_{j+1}) \mu_t^j \quad t \in [1, n], \\
\mu_{t-1}^j - \mu_t^j &\geq 0 \quad j \in [0, K], t \in [1, n] \\
\mu_t^j - \mu_t^{j+1} &\leq 0 \; j \in [0, K-1], \; t \in [1, n], \\
\mu_t^K - \mu_t^0 &\leq 1 \; t \in [1, n], \\
\mu_t^{(k_t-1)} - \zeta_t &\leq \left\lfloor \frac{b_t}{c} \right\rfloor \quad t \in [1, n], \beta_t = f_{k_t}, \\
\mu_t^{\ell_t} - \zeta_t &\geq \left\lfloor \frac{a_t}{c} \right\rfloor + 1 \; t \in [1, n] : \alpha_t = f_{\ell_t} > 0, \\
\mu_t^0 - \zeta_t &\geq \left\lfloor \frac{a_t}{c} \right\rfloor \quad t \in [1, n] : \alpha_t = 0, \\
\zeta_t - \zeta_{t-1} &\leq 0 \quad t \in [1, n], \\
\zeta_t - \zeta_{t-1} &\geq -1 \quad t \in [1, n], \\
\zeta_0 &= 0, \\
\mu_0^i &= 0 \quad i \in [0, K],
\end{aligned}$$

where the interpretation of the μ variables is as follows: $\mu_t^0 = \left\lfloor \frac{\sigma_t}{c} \right\rfloor$, $\mu_t^j = \mu_t^0$ if $\sigma_t - c \left\lfloor \frac{\sigma_t}{c} \right\rfloor < f_j$ and $\mu_t^j = \mu_t^0 + 1$ otherwise.

4.2 Projection of the extended formulation for $Z^{a,b}$

Here $\beta = b \mod c$ and $\alpha = a \mod c$. We present the case in which $\beta > \alpha > 0$. One has $K = 2, (f_0, f_1, f_2, f_3) = (c, \beta, \alpha, 0)$ and the extended formulation becomes:

$$\begin{aligned}
\sigma_t &= (c-\beta)\mu_t^0 + (\beta-\alpha)\mu_t^1 + \alpha\mu_t^2 \;\; t \in [1,n], && (98)\\
\mu_{t-1}^i - \mu_t^i &\geq 0 \;\; i \in [0,2], t \in [1,n], && (99)\\
\mu_t^i - \mu_t^{i+1} &\leq 0 \;\; i \in [0,1], \;\; t \in [1,n], && (100)\\
\mu_t^2 - \mu_t^0 &\leq 1, \;\; t \in [1,n], && (101)\\
\mu_t^0 - \zeta_t &\leq \left\lfloor \frac{b}{c} \right\rfloor \;\; t \in [1,n], && (102)\\
\mu_t^2 - \zeta_t &\geq \left\lfloor \frac{a}{c} \right\rfloor + 1 \;\; t \in [1,n], && (103)\\
\zeta_t - \zeta_{t-1} &\leq 0 \;\; t \in [1,n], && (104)\\
\zeta_t - \zeta_{t-1} &\geq -1 \;\; t \in [1,n], && (105)\\
\zeta_0 &= 0, && (106)\\
\mu_0^i &= 0 \;\; i \in [0,2]. && (107)
\end{aligned}$$

We will now show that the projection on the σ, ζ space gives just the inequalities:

$$\sigma_t \leq \lambda_b \left\lfloor \frac{b}{c} \right\rfloor + \lambda_b \zeta_t \;\; t \in [1,n]$$

and

$$\sigma_t - \sigma_k \leq \lambda_{b-a} \left\lfloor \frac{b-a}{c} \right\rfloor + \lambda_{b-a}(\zeta_t - \zeta_k) \;\; k \in [1,n], t \in [k+1,n],$$

where $\lambda_b = c - \beta$ and $\lambda_{b-a} = c - \beta + \alpha$. In other words,

Theorem 5 conv$(Z^{a,b})$ *with* $\beta > \alpha > 0$ *is described by the constraints (87)-(90),* $z \in [0,1]^n$ *and inequalities*

$$s_t \leq \lambda_b \left\lfloor \frac{b}{c} \right\rfloor + (c - \lambda_b) z_{1t} \;\; t \in [1,n], and$$

$$s_t - s_{k-1} \leq \lambda_{b-a} \left\lfloor \frac{b-a}{c} \right\rfloor + (c - \lambda_{b-a}) z_{kt} \;\; k \in [2,n], t \in [k,n].$$

Proof. We suppose that $\beta > \alpha > 0$. For $k \in [1,n]$, let P_k be the set of vectors that satisfy (98)-(107) for $t \in [1,k]$ and R_k be the set of vectors that satisfy (92)-(97) for $t \in [k+1,n]$ and

$$\sigma_t - \sigma_q \leq (c - \beta + \alpha) \left\lfloor \frac{b-a}{c} \right\rfloor + (c - \beta + \alpha)(\zeta_t - \zeta_q) \;\; q, t \in [k+1,n] : q < t.$$

After elimination of variables for $t \in [k+1,n]$, we obtain $P_k \cap R_k$ plus

$$-(\beta-\alpha)\mu_k^1 - \alpha\mu_k^2 + \sigma_t - (c-\beta)\zeta_t \leq (c-\beta) \left\lfloor \frac{b}{c} \right\rfloor \;\; t \in [k+1,n], \tag{108}$$

$$-(\beta-\alpha)\mu_k^1 + \sigma_t - (c-\beta+\alpha)\zeta_t \leq (c-\beta+\alpha) \left\lfloor \frac{b}{c} \right\rfloor + \alpha \;\; t \in [k+1,n]. \tag{109}$$

For $k = n$, we have P_n, which is the original system. When $k = 0$, we obtain R_0 and inequalities (108) and (109). We now show that the inequalities (109) are redundant. For $k = 0$ and $t \in [1, n]$, inequality (109) is

$$\sigma_t - (c - \beta + \alpha)\zeta_t \leq (c - \beta + \alpha) \left\lfloor \frac{b}{c} \right\rfloor + \alpha.$$

This can be obtained by taking $\sigma_t - (c-\beta)\zeta_t \leq \left\lfloor \frac{b}{c} \right\rfloor (c-\beta)$ with weight $(\beta-\alpha)/\beta$ and $\sigma_t - c\zeta_t \leq b$ with weight α/β. We give the induction proof for the intermediate iterations in Appendix 2. □

The other cases in which $\alpha = 0$ or $\alpha \geq \beta \geq 0$ lead to the same result. However σ_t, k_t and ℓ_t are modified in the extended formulation and thus the projection also needs to be modified.

Observation 3 *If one modifies the set Z by replacing the single entering arc by multiple entering arcs with associated 0-1 variables z_t^j and sets $z_t = \sum_{j=1}^{J} z_t^j$, the new variables again give rise to dual network constraints, so that to obtain the projection of this new set, it suffices to remove the bounds on z_t, substitute for z_t and add the bound constraints $0 \leq z_t^j \leq 1$ for $j \in [1, J], t \in [1, T]$.*

5 Final Remarks

Most mixed integer programming solvers are nowadays very good at generating flow cover inequalities so it is doubtful whether the inequalities and extended formulations presented here can have much computational impact. In particular it turns out that the lot-sizing problem with constraint set Y that is not demand driven is easily solved computationally. We see this work rather as part of a continuing attempt to analyze and understand simple mixed integer programs. It also confirms that, for certain tight extended formulations, Fourier-Motzkin elimination, though tedious and far from elegant, can be used to carry out non-trivial projections. The conjecture listed in the Introduction remains a challenge as we have not been able to find a tight extended formulation. It may also be useful to develop other valid inequalities for the cases in which the constant capacity flow cover inequalities do not suffice to give the convex hull.

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Appendix 1: Proof of Theorem 4

The proof is by induction. The result holds for $k = n$. We show that if it holds for k, then it holds for $k - 1$.

For ease of presentation, we introduce $\pi_u = \min\{x_u, (c - \lambda)z_u\}$ for $u \in [1, n]$.

Elimination of α_k^2 and σ_k^2

The first step is to use the equations (75) and (76) for $t = k$ to eliminate α_k^2 and σ_k^2 by substitution.

Inequality (79) for $t = k$ becomes

$$s_k \leq (c - \lambda)(\sigma_{k-1}^1 + \sigma_{k-1}^2) + x_k - \lambda\alpha_k^1 + \lambda\sigma_k^1. \tag{110}$$

Inequality (80) for $t = k$ becomes

$$x_k \leq (c - \lambda)z_k + \lambda\alpha_k^1. \tag{111}$$

Inequality (81) for $t = k$ becomes

$$x_k - c\alpha_k^1 \leq s_k - c\sigma_k^1. \tag{112}$$

Inequality (83) for $t = k$ becomes

$$s_k \leq c\sigma_k^1 + (c - \lambda). \tag{113}$$

Inequality (84) for $t = k$ becomes

$$x_k \geq c\alpha_k^1. \tag{114}$$

Inequalities (74) for $t \in [k + 1, n]$ become

$$s_t \leq \lambda \left\lfloor \frac{b_t}{c} \right\rfloor + \pi_{k+1,t} - \lambda\sigma_k^1 + s_k. \tag{115}$$

Elimination of α_k^1

Now we use Fourier-Motzkin to eliminate α_k^1. We have to combine the inequalities (110) and (114) with the inequalities (78), (85), (111) and (112).
Inequality (78) with (110) gives

$$s_k \leq x_k + s_{k-1}.$$

Inequality (78) with (114) gives

$$x_k \geq c(\sigma^1_k - \sigma^1_{k-1}). \tag{116}$$

Inequality (85) with (114) gives

$$x_k \geq 0.$$

Inequality (85) with (110) gives

$$s_k \leq (c-\lambda)(\sigma^1_{k-1} + \sigma^2_{k-1}) + x_k + \lambda\sigma^1_k. \tag{117}$$

Inequality (111) with (110) gives

$$s_k \leq (c-\lambda)z_k + (c-\lambda)(\sigma^1_{k-1} + \sigma^2_{k-1}) + \lambda\sigma^1_k. \tag{118}$$

These inequalities (117) and (118) together give

$$s_k \leq \pi_k + (c-\lambda)(\sigma^1_{k-1} + \sigma^2_{k-1}) + \lambda\sigma^1_k. \tag{119}$$

Inequality (111) with (114) gives

$$x_k \leq cz_k.$$

Inequality (112) with (110) gives

$$s_k \leq x_k + c(\sigma^1_{k-1} + \sigma^2_{k-1}). \tag{120}$$

Inequality (112) with (114) gives

$$s_k \geq c\sigma^1_k. \tag{121}$$

Note that inequality (120) is redundant as it is dominated by (116).

Elimination of σ^1_k

σ^1_k occurs in the inequalities (82), (115), (116), (121) and with the opposite sign in (84), (113), (119). We treat inequalities (82) and (115) together as (82) is the same as (115) for $t = k$.

Combining (84) with (121) gives

$$s_k \geq 0.$$

Combining (113) with (115) for $t \in [k, n]$ gives

$$cs_t - (c-\lambda)s_k \leq \lambda b_t + c\pi_{k+1,t}.$$

For $t = k$, we obtain

$$s_k \leq b_k.$$

For $t \in [k+1, n]$, these inequalities are redundant. To see this, we take inequality $s_u - s_{u-1} \leq x_u$ for $[u \in k+1, t]$ with weight $(c-\lambda)$ and $s_t \leq b_t$ with weight λ. We obtain $cs_t - (c-\lambda)s_k \leq \lambda b_t + (c-\lambda)x_{k+1,t}$. Now $(c-\lambda)x_u \leq cx_u$ and as $x_u \leq cz_u$, $(c-\lambda)x_u \leq c(c-\lambda)z_u$. Thus

$(c-\lambda)x_u \le c\pi_u$ for $u \in [k+1,t]$ and the inequality is dominated.
Combining (119) with (115) for $t \in [k,n]$ gives

$$s_t \le \lambda \left\lfloor \frac{b_t}{c} \right\rfloor + \pi_{kt} + (c-\lambda)(\sigma^1_{k-1} + \sigma^2_{k-1}) \quad t \in [k,n].$$

Note that these are inequalities (74) of $R^{k-1} \setminus R^k$.
Combining (84) with (115) for $t \in [k,n]$ gives

$$s_t - s_k \le \left\lfloor \frac{b_t}{c} \right\rfloor \lambda + \pi_{k+1,t}.$$

The inequality is redundant for $t = k$. For $t \in [k+1,n]$, we obtain the inequalities (73) for $j = k$ of $R^{k-1} \setminus R^k$.
Note that we have obtained all the inequalities describing R^{k-1}.

It now suffices to show that the remaining inequalities are redundant for $Q^{k-1} \cap R^{k-1}$.
Combining (84) with (82) and (116) give $\left\lfloor \frac{b_k}{c} \right\rfloor \ge 0$ and $x_k + c\sigma^1_{k-1} \ge 0$, respectively. These are both redundant.
Combining (113) with (121) gives $c - \lambda \ge 0$, which is redundant.
Combining (113) with (116) gives

$$s_k \le x_k + c\sigma^1_{k-1} + (c-\lambda).$$

Combining (119) with (121) gives

$$(c-\lambda)s_k \le c\pi_k + c(c-\lambda)(\sigma^1_{k-1} + \sigma^2_{k-1}).$$

Combining (119) with (116) gives

$$cs_k \le (c+\lambda)\pi_k + cs_{k-1}.$$

The last three inequalities are all dominated using (116), $s_k \le x_k + s_{k-1}$ and (120), $x_k \le cz_k$ together with $\pi_k = \min\{x_k, (c-\lambda)z_k\}$ and $s_{k-1} = c\sigma^1_{k-1} + (c-\lambda)\sigma^2_{k-1}$.

Now the result follows by induction as $R^0 \cap Q^0$ is as claimed in the Theorem. □

Appendix 2: Proof of Theorem 5

Suppose that after the elimination of variables for $t \in [k+1,n]$, we obtain $P_k \cap R_k$ plus inequalities (108) and (109) for $t \in [k+1,n]$. We will show that after eliminating μ^0_k, μ^1_k and μ^2_k, we obtain $P_{k-1} \cap R_{k-1}$ plus inequalities (108) and (109) for $t \in [k,n]$.

Elimination of μ^0_k

The first step is the elimination of μ^0_k by substitution. The inequalities $\mu^0_k \le \mu^0_{k-1}$, $\mu^0_k \le \mu^1_k$, $\mu^0_k - \zeta_k \le \left\lfloor \frac{b}{c} \right\rfloor$ and $\mu^2_k - \mu^0_k \le 1$ give

$$\sigma_k \le (c-\beta)\mu^0_{k-1} + (\beta-\alpha)\mu^1_k + \alpha\mu^2_k, \tag{122}$$

$$\sigma_k \le (c-\alpha)\mu^1_k + \alpha\mu^2_k, \tag{123}$$

$$\sigma_k - (c-\beta)\zeta_k \le (c-\beta)\left\lfloor \frac{b}{c} \right\rfloor + (\beta-\alpha)\mu^1_k + \alpha\mu^2_k, \tag{124}$$

$$-\sigma_k + (\beta-\alpha)\mu^1_k + (c-\beta+\alpha)\mu^2_k \le (c-\beta), \tag{125}$$

respectively.

Elimination of μ_k^1

Now we eliminate μ_k^1 using Fourier-Motzkin elimination. Overall, after eliminating μ_k^0, we are left with $P_{k-1} \cap R_k$ plus (99), (100), (103), (108), (109), (122), (123), (124) and (125).

The constraints involving μ_k^1 are all these inequalities except (99) for $i = 2$ and (103). As (124) is the same as (108) for $t = k$, we treat them together.

The resulting inequalities are:
(99) and (108) for $t \in [k, n]$ give

$$(\beta - \alpha)\mu_{k-1}^1 \geq \sigma_t - (c - \beta)(\zeta_t + \left\lfloor \frac{b}{c} \right\rfloor) - \alpha\mu_k^2 \quad t \in [k, n]. \tag{126}$$

(99) and (109) for $t \in [k+1, n]$ give

$$(\beta - \alpha)\mu_{k-1}^1 \geq \sigma_t - (c - \beta + \alpha)(\zeta_t + \left\lfloor \frac{b}{c} \right\rfloor) - \alpha \quad t \in [k+1, n]. \tag{127}$$

(99) and (122) give

$$(\beta - \alpha)\mu_{k-1}^1 \geq \sigma_k - \alpha\mu_k^2 - (c - \beta)\mu_{k-1}^0. \tag{128}$$

(99) and (123) give $(c - \alpha)\mu_{k-1}^1 \geq \sigma_k - \alpha\mu_k^2$. This is redundant since adding (128) and $(c - \beta)$ times $\mu_{k-1}^1 \geq \mu_{k-1}^0$ gives this inequality.

(100) and (108) for $t \in [k, n]$ give

$$\beta\mu_k^2 \geq \sigma_t - (c - \beta)(\left\lfloor \frac{b}{c} \right\rfloor + \zeta_t) \quad t \in [k, n]. \tag{129}$$

(100) and (109) give $(\beta - \alpha)\mu_k^2 \geq \sigma_t - (c - \beta + \alpha)(\zeta_t + \lfloor \frac{b}{c} \rfloor) - \alpha \quad t \in [k+1, n]$. This is redundant as it can be obtained by taking $b \geq \sigma_t - c\zeta_t$ with weight $\frac{\alpha}{\beta}$ and (129) for t with weight $\frac{\beta - \alpha}{\beta}$.

(100) and (122) give

$$\beta\mu_k^2 \geq \sigma_k - (c - \beta)\mu_{k-1}^0. \tag{130}$$

(100) and (123) give

$$c\mu_k^2 \geq \sigma_k. \tag{131}$$

(125) and (108) for $t \in [k, n]$ give

$$\sigma_k - (c - \beta)\mu_k^2 + (c - \beta)(\left\lfloor \frac{b}{c} \right\rfloor + 1 + \zeta_t) \geq \sigma_t \quad t \in [k, n].$$

For $t \in [k+1, n]$, the inequality is redundant since adding $\frac{c - \beta}{c - \beta + \alpha}$ times (133) (obtained from (125) and (109) below) and $\frac{\alpha}{c - \beta + \alpha}$ times $\sigma_k \geq \sigma_t$ gives this inequality. For $t = k$, we obtain

$$\mu_k^2 - \zeta_k \leq 1 + \left\lfloor \frac{b}{c} \right\rfloor. \tag{132}$$

(125) and (109) give

$$\sigma_k - (c-\beta+\alpha)\mu_k^2 + (c-\beta+\alpha)(\left\lfloor \frac{b}{c} \right\rfloor + 1 + \zeta_t) \geq \sigma_t \quad t \in [k+1, n]. \tag{133}$$

(125) and (122) give $\mu_k^2 \leq \mu_{k-1}^0 + 1$. This is redundant since $\mu_k^2 \leq \mu_{k-1}^2 \leq \mu_{k-1}^0 + 1$.

(125) and (123) give

$$-\sigma_k + c\mu_k^2 \leq c - \alpha. \tag{134}$$

The resulting formulation is $P_{k-1} \cap R_k$ plus inequalities

$$(127) \quad \sigma_t - (c-\beta+\alpha)\zeta_t - (\beta-\alpha)\mu_{k-1}^1 \leq (c-\beta+\alpha)\left\lfloor \frac{b}{c} \right\rfloor + \alpha \quad t \in [k+1, n]$$

that do not involve μ_k^2 and the following families of inequalities that involve μ_k^2:
(99) for $i = 2$, (132), (134), (133), (131), (103), (130), (128), (129) and (126).

Elimination of μ_k^2

Here we need to combine the four inequalities (99) for $i = 2$, (132), (134) and (133) with the six inequalities (131), (103), (130), (128), (129) and (126) to eliminate μ_k^2.
Combining (99) and (128) gives (92) for $t = k$:

$$\sigma_k - \sigma_{k-1} \leq 0.$$

Combining (99) and (126) gives for $t \in [k, n]$:

$$\sigma_t - (c-\beta)\zeta_t - (\beta-\alpha)\mu_{k-1}^1 - \alpha\mu_{k-1}^2 \leq (c-\beta)\left\lfloor \frac{b}{c} \right\rfloor.$$

These are inequalities (108) for $t \in [k, n]$.

Combining (132)and (129) for $t = k$ gives:

$$\sigma_k - c\zeta_k \leq b,$$

which is (93) for $t = k$.
Combining (132) and (126) for $t = k$ gives

$$\sigma_k - (c-\beta+\alpha)\zeta_k - (\beta-\alpha)\mu_{k-1}^1 \leq (c-\beta+\alpha)\left\lfloor \frac{b}{c} \right\rfloor + \alpha.$$

This is (109) for $t = k$. Together with (127), we have (109) for $t \in [k, n]$.

Combining (134) and (103) gives (94) for $t = k$:

$$-\sigma_k + c\zeta_k \leq -a.$$

Combining (133) and (103) gives for $t \in [k+1, n]$

$$\sigma_t - \sigma_k - (c-\beta+\alpha)(\zeta_t - \zeta_k) \leq (c-\beta+\alpha)(\left\lfloor \frac{b}{c} \right\rfloor - \left\lfloor \frac{a}{c} \right\rfloor).$$

These are inequalities (108) since $\left\lfloor \frac{b-a}{c} \right\rfloor = \left\lfloor \frac{b}{c} \right\rfloor - \left\lfloor \frac{a}{c} \right\rfloor$.

The remaining combinations are redundant:

Combining (99)) and (131) gives

$$\sigma_k - c\mu^2_{k-1} \leq 0.$$

It is redundant taking $\sigma_{k-1} - (c-\beta)\mu^0_{k-1} - (\beta-\alpha)\mu^1_{k-1} - \alpha\mu^2_{k-1} = 0$ with weight 1, $\mu^0_{k-1} - \mu^1_{k-1} \leq 0$ with weight $(c-\beta)$, $\mu^1_{k-1} - \mu^2_{k-1} \leq 0$ with weight $(c-\alpha)$ and $-\sigma_{k-1} + \sigma_k \leq 0$ with weight 1.

Combining (99) and (103) gives

$$-\mu^2_{k-1} + \zeta_k \leq -\left\lfloor \frac{a}{c} \right\rfloor - 1.$$

It is redundant taking $-\zeta_{k-1} + \zeta_k \leq 0$ with weight 1 and $\zeta_{k-1} - \mu^2_{k-1} \leq -\left\lfloor \frac{a}{c} \right\rfloor - 1$ with weight 1.

Combining (99) and (130) gives

$$\sigma_k - (c-\beta)\mu^0_{k-1} - \beta\mu^2_{k-1} \leq 0.$$

It is redundant taking $\sigma_{k-1} - (c-\beta)\mu^0_{k-1} - (\beta-\alpha)\mu^1_{k-1} - \alpha\mu^2_{k-1} = 0$ with weight 1, $\mu^1_{k-1} - \mu^2_{k-1} \leq 0$ with weight $(\beta-\alpha)$ and $-\sigma_{k-1} + \sigma_k \leq 0$ with weight 1.

Combining (99) and (129) for $t \in [k, n]$ gives

$$\sigma_t - (c-\beta)\zeta_t - \beta\mu^2_{k-1} \leq (c-\beta)\left\lfloor \frac{b}{c} \right\rfloor.$$

It is redundant taking (108) $\sigma_t - (c-\beta)\zeta_t - (\beta-\alpha)\mu^1_{k-1} - \alpha\mu^2_{k-1} \leq (c-\beta)\left\lfloor \frac{b}{c} \right\rfloor$ with weight 1 and $\mu^1_{k-1} - \mu^2_{k-1} \leq 0$ with weight $(\beta-\alpha)$.

Combining (132) and (131)) gives

$$\sigma_k - c\zeta_k \leq c(1 + \left\lfloor \frac{b}{c} \right\rfloor).$$

This is redundant taking $\sigma_k - c\zeta_k \leq b$.

Combining (132) and (103) gives

$$0 \leq \left\lfloor \frac{b}{c} \right\rfloor - \left\lfloor \frac{a}{c} \right\rfloor.$$

This is redundant as $a \leq b$.

Combining (132) and (130) gives

$$\sigma_k - (c-\beta)\mu^0_{k-1} - \beta\zeta_k \leq \beta(1 + \left\lfloor \frac{b}{c} \right\rfloor).$$

This is redundant taking $\sigma_{k-1} - (c-\beta)\mu^0_{k-1} - (\beta-\alpha)\mu^1_{k-1} - \alpha\mu^2_{k-1} = 0$ with weight $\frac{c-\beta}{c}$, $\sigma_k - c\zeta_k \leq b$ with weight $\frac{\beta}{c}$, $\mu^1_{k-1} - \mu^2_{k-1} \leq 0$ with weight $\frac{c-\beta}{c}(\beta-\alpha)$, $\mu^2_{k-1} - \mu^0_{k-1} \leq 1$ with weight $\frac{\beta}{c}(c-\beta)$ and $-\sigma_{k-1} + \sigma_k \leq 0$ with weight $\frac{c-\beta}{c}$.

Combining (132) and (128) gives

$$\sigma_k - (c-\beta)\mu^0_{k-1} - (\beta-\alpha)\mu^1_{k-1} - \alpha\zeta_k \leq \alpha(1 + \left\lfloor \frac{b}{c} \right\rfloor).$$

This is redundant taking $-(\beta-\alpha)\mu^1_{k-1} + \sigma_k - (c-\beta+\alpha)\zeta_k \leq (c-\beta+\alpha)\left\lfloor \frac{b}{c} \right\rfloor + \alpha$ with weight $\frac{\alpha}{c-\beta+\alpha}$, $\sigma_{k-1} - (c-\beta)\mu^0_{k-1} - (\beta-\alpha)\mu^1_{k-1} - \alpha\mu^2_{k-1} = 0$ with weight $\frac{c-\beta}{c-\beta+\alpha}$, $\mu^2_{k-1} - \mu^0_{k-1} \leq 1$ with weight $\frac{\alpha(c-\beta)}{c-\beta+\alpha}$ and $-\sigma_{k-1} + \sigma_k \leq 0$ with weight $\frac{c-\beta}{c-\beta+\alpha}$.

Combining (132) and (129) for $t \in [k+1, n]$ gives

$$\sigma_t - (c-\beta)\zeta_t - \beta\zeta_k \leq b.$$

This is redundant taking $\sigma_k - c\zeta_k \leq b$ with weight $\frac{\beta}{c}$, $\sigma_t - c\zeta_t \leq b$ with weight $\frac{c-\beta}{c}$ and $-\sigma_{j-1} + \sigma_j \leq 0$ with weight $\frac{\beta}{c}$ for $j \in [k+1, t]$.

Combining (132) and (126) for $t \in [k+1, n]$ gives

$$\sigma_t - (c-\beta)\zeta_t - (\beta-\alpha)\mu^1_{k-1} - \alpha\zeta_k \leq (c-\beta+\alpha)\left\lfloor \frac{b}{c} \right\rfloor + \alpha.$$

This is redundant taking ((109) for k) $-(\beta-\alpha)\mu^1_{k-1} + \sigma_k - (c-\beta+\alpha)\zeta_k \leq (c-\beta+\alpha)\left\lfloor \frac{b}{c} \right\rfloor + \alpha$ with weight $\frac{\alpha}{c-\beta+\alpha}$, ((109) for t) $-(\beta-\alpha)\mu^1_{k-1} + \sigma_t - (c-\beta+\alpha)\zeta_t \leq (c-\beta+\alpha)\left\lfloor \frac{b}{c} \right\rfloor + \alpha$ with weight $1 - \frac{\alpha}{c-\beta+\alpha}$ and $-\sigma_{j-1} + \sigma_j \leq 0$ with weight$\frac{\alpha}{c-\beta+\alpha}$ for $j \in [k+1, t]$.

Combining (134) and (131) gives

$$c - \alpha \geq 0$$

which is redundant.

Combining (134) and (130) gives

$$(c-\beta)\sigma_k - (c-\beta)c\mu^0_{k-1} \leq (c-\alpha)\beta.$$

This is redundant taking $\sigma_{k-1} - (c-\beta)\mu^0_{k-1} - (\beta-\alpha)\mu^1_{k-1} - \alpha\mu^2_{k-1} = 0$ with weight $(c-\beta)$, $\mu^1_{k-1} - \mu^2_{k-1} \leq 0$ with weight $(c-\beta)(\beta-\alpha)$, $\mu^2_{k-1} - \mu^0_{k-1} \leq 1$ with weight $(c-\beta)\beta$, $-\sigma_{k-1} + \sigma_k \leq 0$ with weight $(c-\beta)$and $(c-\beta)\beta \leq (c-\alpha)\beta$ with weight 1.

Combining (134) and (128) gives

$$(c-\alpha)\sigma_k - c(c-\beta)\mu^0_{k-1} - c(\beta-\alpha)\mu^1_{k-1} \leq \alpha(c-\alpha).$$

This is redundant taking $\sigma_{k-1} - (c-\beta)\mu^0_{k-1} - (\beta-\alpha)\mu^1_{k-1} - \alpha\mu^2_{k-1} = 0$ with weight $(c-\alpha)$, $\mu^0_{k-1} - \mu^1_{k-1} \leq 0$ with weight $\alpha(\beta-\alpha)$, $\mu^2_{k-1} - \mu^0_{k-1} \leq 1$ with weight $\alpha(c-\alpha)$ and $-\sigma_{k-1} + \sigma_k \leq 0$ with weight $(c-\alpha)$.

Combining (134) and (129) for $t \in [k, n]$ gives

$$c\sigma_t - c(c-\beta)\zeta_t - \beta\sigma_k \leq \beta(c-\alpha) + (c-\beta)c\left\lfloor \frac{b}{c} \right\rfloor.$$

This is redundant taking $\sigma_t - c\zeta_t \leq b$ with weight $(c-\beta)$, $-\sigma_{j-1} + \sigma_j \leq 0$ with weight β for $j \in [k+1, t]$, $b(c-\beta) = \beta(c-\beta) + (c-\beta)c\left\lfloor \frac{b}{c} \right\rfloor$ with weight 1 and $\beta(c-\beta) \leq \beta(c-\alpha)$ with weight 1.

Combining (134) and (126) for $t \in [k, n]$ gives

$$c\sigma_t - c(c-\beta)\zeta_t - \alpha\sigma_k - c(\beta-\alpha)\mu^1_{k-1} \leq c(c-\beta)\left\lfloor \frac{b}{c} \right\rfloor + (c-\alpha)\alpha.$$

This is redundant taking
((109) for t) $\sigma_t - (c-\beta+\alpha)\zeta_t - (\beta-\alpha)\mu^1_{k-1} \leq (c-\beta+\alpha)\left\lfloor \frac{b}{c} \right\rfloor + \alpha$ with weight $\frac{c(c-\beta)}{c-\beta+\alpha}$, $\sigma_{k-1} - (c-\beta)\mu^0_{k-1} - (\beta-\alpha)\mu^1_{k-1} - \alpha\mu^2_{k-1} = 0$ with weight $\frac{\alpha(\beta-\alpha)}{c-\beta+\alpha}$, $\mu^0_{k-1} - \mu^1_{k-1} \leq 0$ with weight $\alpha(\beta-\alpha)$, $\mu^2_{k-1} - \mu^0_{k-1} \leq 1$ with weight $\frac{\alpha^2(\beta-\alpha)}{c-\beta+\alpha}$, $-\sigma_{k-1} + \sigma_k \leq 0$ with weight $\frac{\alpha(\beta-\alpha)}{c-\beta+\alpha}$ and $-\sigma_{j-1} + \sigma_j \leq 0$ with weight $\frac{\alpha c}{c-\beta+\alpha}$ for $j \in [k+1, t]$.

Combining (133) for $t \in [k+1, n]$ and ((131) gives

$$c\sigma_t - c(c-\beta+\alpha)\zeta_t - (\beta-\alpha)\sigma_k \leq c(c-\beta+\alpha)(\left\lfloor \frac{b}{c} \right\rfloor + 1).$$

This is redundant taking $-\sigma_{j-1} + \sigma_j \leq 0$ with weight $(\beta-\alpha)$ for $j \in [k+1, t]$, $\sigma_t - c\zeta_t \leq b$ with weight $(c-\beta+\alpha)$ and $\beta \leq c$ with weight $c-\beta+\alpha$.

Combining (133) for $t \in [k+1, n]$ and (130) gives

$$\beta\sigma_t - \beta(c-\beta+\alpha)\zeta_t + (c-2\beta+\alpha)\sigma_k - (c-\beta+\alpha)(c-\beta)\mu^0_{k-1} \leq \beta(c-\beta+\alpha)(\left\lfloor \frac{b}{c} \right\rfloor + 1).$$

case 1. $c - 2\beta + \alpha \geq 0$
This is redundant taking ((109) for t) $\sigma_t - (c-\beta+\alpha)\zeta_t - (\beta-\alpha)\mu^1_{k-1} \leq (c-\beta+\alpha)\left\lfloor \frac{b}{c} \right\rfloor + \alpha$ with weight β, $\sigma_{k-1} - (c-\beta)\mu^0_{k-1} - (\beta-\alpha)\mu^1_{k-1} - \alpha\mu^2_{k-1} = 0$ with weight $c-2\beta+\alpha$, $\mu^1_{k-1} - \mu^2_{k-1} \leq 0$ with weight $(\beta-\alpha)(c-\beta+\alpha)$, $-\mu^0_{k-1} + \mu^2_{k-1} \leq 1$ with weight $\beta(c-\beta)$ and $-\sigma_{k-1} + \sigma_k \leq 0$ with weight $c-2\beta+\alpha$.
case 2. $c - 2\beta + \alpha < 0$
This is redundant taking $\sigma_t - c\zeta_t \leq b$ with weight $\frac{(c-\beta+\alpha)(2\beta-c-\alpha)}{\beta-\alpha}$, $\sigma_t - (c-\beta+\alpha)\zeta_t - (\beta-\alpha)\mu^1_{k-1} \leq (c-\beta+\alpha)\left\lfloor \frac{b}{c} \right\rfloor + \alpha$ with weight $\frac{(c-\beta+\alpha)(c-\beta)}{\beta-\alpha}$, $\mu^1_{k-1} - \mu^2_{k-1} \leq 0$ with weight $(c-\beta)(c-\beta+\alpha)$, $-\mu^0_{k-1} + \mu^2_{k-1} \leq 1$ with weight $(c-\beta)(c-\beta+\alpha)$and $-\sigma_{j-1} + \sigma_j \leq 0$ for $j \in [k+1, t]$ with weight $2\beta - c - \alpha$.

Combining (133) for $t \in [k+1, n]$ and (128) gives

$$\alpha\sigma_t + (c-\beta)\sigma_k - (c-\beta+\alpha)(\alpha\zeta_t + (c-\beta)\mu^0_{k-1} + (\beta-\alpha)\mu^1_{k-1}) \leq \alpha(c-\beta+\alpha)(\left\lfloor \frac{b}{c} \right\rfloor + 1).$$

This is redundant taking ((109) for t) $-(\beta-\alpha)\mu^1_{k-1} + \sigma_t - (c-\beta+\alpha)\zeta_t \leq (c-\beta+\alpha)\left\lfloor \frac{b}{c} \right\rfloor + \alpha$ with weight α, $\sigma_{k-1} - (c-\beta)\mu^0_{k-1} - (\beta-\alpha)\mu^1_{k-1} - \alpha\mu^2_{k-1} = 0$ with weight $(c-\beta)$, $\mu^2_{k-1} - \mu^0_{k-1} \leq 1$

with weight $\alpha(c-\beta)$ and $-\sigma_{k-1}+\sigma_k \le 0$ with weight $(c-\beta)$.

Combining (133) for $t \in [k+1,n]$ and (129) for $q \in [k,n]$ gives

$$\beta[\sigma_t-(c-\beta+\alpha)\zeta_t]+(c-\beta+\alpha)[\sigma_q-(c-\beta)\zeta_q]-\beta\sigma_k \le (c-\beta+\alpha)b.$$

case 1. $t>q$
This is redundant taking $\sigma_t - c\zeta_t \le b$ with weight $(c-\beta+\alpha)\beta/c$, $\sigma_q - c\zeta_q \le b$ with weight $(c-\beta+\alpha)(c-\beta)/c$, $-\sigma_{j-1}+\sigma_j \le 0$ with weight β for $j \in [k+1,q]$ and $-\sigma_{j-1}+\sigma_j \le 0$ with weight $\beta(\beta-\alpha)/c$ for $j \in [q+1,t]$.
case 2. $t \le q$
This is redundant taking $\sigma_t - c\zeta_t \le b$ with weight $(c-\beta+\alpha)\beta/c$, $\sigma_q - c\zeta_q \le b$ with weight $(c-\beta+\alpha)(c-\beta)/c$, $-\sigma_{j-1}+\sigma_j \le 0$ with weight β for $j \in [k+1,t]$ and $-\sigma_{j-1}+\sigma_j \le 0$ with weight $\beta(c-\beta+\alpha)/c$ for $j \in [t+1,q]$.

Combining (133) for $t \in [k+1,n]$ and (126) for $q \in [k,n]$ gives

$$\alpha[\sigma_t-(c-\beta+\alpha)\zeta_t]+(c-\beta+\alpha)[\sigma_q-(c-\beta)\zeta_q]-\alpha\sigma_k-(c-\beta+\alpha)(\beta-\alpha)\mu^1_{k-1} \le (c-\beta+\alpha)[(c-\beta+\alpha)\left\lfloor \frac{b}{c} \right\rfloor+\alpha].$$

This is redundant taking ((109) for t) $-(\beta-\alpha)\mu^1_{k-1}+\sigma_t-(c-\beta+\alpha)\zeta_t \le (c-\beta+\alpha)\left\lfloor \frac{b}{c} \right\rfloor+\alpha$ with weight α, ((109) for q) $-(\beta-\alpha)\mu^1_{k-1}+\sigma_q-(c-\beta+\alpha)\zeta_q \le (c-\beta+\alpha)\left\lfloor \frac{b}{c} \right\rfloor+\alpha$ with weight $(c-\beta)$ and $-\sigma_{j-1}+\sigma_j \le 0$ with weight α for $j \in [k+1,q]$.

The iteration is complete and we obtain the formulation with k replaced by $k-1$. $\square$