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Convex Hull Results for the Warehouse Problem

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Abstract

Given an initial stock and a capacitated warehouse, the warehouse problem aims to decide when to sell and purchase to maximize profit. This problem is common in revenue management and energy storage. We extend this problem by incorporating fixed costs and provide convex hull descriptions as well as tight compact extended formulations for several variants. For this purpose, we first derive unit flow formulations based on characterizations of extreme points and then project out the additional variables using Fourier-Motzkin elimination. It turns out that the nontrivial inequalities are flow cover inequalities for some single node flow set relaxations.

Keywords: warehouse problem, integer programming, convex hull, extended formulation, Fourier-Motzkin elimination, single node flow set, flow cover inequalities..
AMS 2010 Mathematics Subject Classification: 90C27, 90C57

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1. Introduction

The warehouse problem, introduced by Cahn [1948], is to optimally decide on purchasing (or production), storage and sales quantities for a product with a fixed warehouse capacity and a given initial stock. This is a common problem in storage and revenue management in a commodity market. A commodity is a raw material or an agricultural product such as grains, vegetables, coal and natural gas.

The formal definition of the basic problem is as follows. Suppose that the initial stock is S units and the warehouse has a capacity of B units. We are given a planning horizon of n periods. The buying price is c_t and the selling price is p_t in period t. In each period, we can sell at most as much as the inventory from the previous period, i.e., the amount purchased in a period cannot be sold in the same period. The aim of the warehouse problem is to decide on how much to purchase and sell in each period to maximize the total profit.

We define x_t to be the amount purchased and y_t to be the amount sold in period t. For two integers $n_1 \leq n_2$, we let $[n_1, n_2] = \{n_1, \ldots, n_2\}$. For a vector $a \in \mathbb{R}^n$, we use $a_{ut} = \sum_{i=u}^{t} a_i$ and $a_T = \sum_{i \in T} a_i$ for $T \subseteq [1, n]$. The basic warehouse problem can be modeled as

$$
\begin{aligned}
\max \sum_{t=1}^{n} & (p_t y_t - c_t x_t) \\
\text{s.t. } & S + x_{1t} - y_{1t} \leq B \qquad t \in [1, n], \\
& y_t \leq S + x_{1,t-1} - y_{1,t-1} \qquad t \in [1, n], \\
& x_t, y_t \geq 0 \qquad t \in [1, n].
\end{aligned}
$$

Note here that $S + x_{1t} - y_{1t}$ is the amount of stock at the end of period t. The first and second sets of constraints ensure that the stock does not exceed the warehouse capacity and that we cannot sell more than what is available in stock from the previous period, respectively. The objective function is equal to the revenue minus purchasing cost.

Charnes and Cooper [1955] generalize this problem to the case of multiple products and varying prices. Bellman [1956] presents a dynamic programming algorithm. Dreyfus [1957] shows that the solution can be determined analytically. He shows that there are four policies: sell all the stock, buy up to capacity, sell and buy and do nothing. Consequently, an optimal policy is to do nothing for a number of stages and then alternate between a full and empty warehouse. Eastman [1959] models the problem as a shortest path problem. Charnes et al. [1959] use the warehouse model to illustrate how linear programming can be used for allocation of funds in an entreprise. The multistage stochastic warehouse problem is studied by Charnes et al. [1966].

Many more complex and mostly stochastic variants of the warehouse problem have been studied in the context of optimal commodity trading and energy storage (see, e.g., Devalkar et al. [2011], Harsha and Dahleh [2015], Secomandi [2008], Secomandi [2010], Wu et al. [2012], Zhou et al. [2015]). However, to the best of our knowledge, there is no study on strong formulations of this problem in the presence of fixed costs. In this study, we extend the warehouse problem by including a fixed cost for buying and/or selling and inventory holding costs. We provide convex hull descriptions and tight compact extended formulations.

Let h_t denote the inventory holding cost, f_t and g_t denote the fixed costs for buying and selling, respectively, in period t. In addition to the variables defined above, we define s_t to be the amount of stock at the end of period t and the binary variables z_t and w_t to be 1 if we buy and sell in period t and 0 otherwise, respectively.

We study three variants of the warehouse problem:

- WP1: In the first variant, we include fixed costs only for buying. This variant can be modeled as:

$$
\begin{aligned}
\max & \sum_{t=1}^{n} (p_t y_t - c_t x_t - f_t z_t - h_t s_t) \\
\text{s.t. } & s_0 = S, && (1) \\
& s_{t-1} + x_t = y_t + s_t \quad t \in [1, n], && (2) \\
& 0 \le y_t \le s_{t-1} \quad t \in [1, n], && (3) \\
& 0 \le s_t \le B \quad t \in [1, n], && (4) \\
& 0 \le x_t \le B z_t \quad t \in [1, n], && (5) \\
& z_t \in \{0, 1\} \quad t \in [1, n]. && (6)
\end{aligned}
$$

- WP2: In the second variant, we have fixed costs both for buying and selling.

$$
\begin{aligned}
\max & \sum_{t=1}^{n} (p_t y_t - g_t w_t - c_t x_t - f_t z_t - h_t s_t) \\
\text{s.t. } & (1)\text{-}(6), \\
& y_t \le B w_t \quad t \in [1, n], && (7) \\
& w_t \in \{0, 1\} \quad t \in [1, n]. && (8)
\end{aligned}
$$

- WP3: In the third variant, we have fixed costs both for buying and selling and we do not allow to buy and sell in the same period.

$$
\begin{aligned}
\max & \sum_{t=1}^{n} (p_t y_t - g_t w_t - c_t x_t - f_t z_t - h_t s_t) \\
\text{s.t. } & (1)\text{-}(8), \\
& w_t + z_t \le 1 \quad t \in [1, n]. && (9)
\end{aligned}
$$

It is possible to model all three problems without the stock variables. For the first variant, the resulting feasible set, denoted X_1, is:

$$
\begin{aligned}
& y_{1t} \le S + x_{1,t-1} \quad t \in [1, n], && (10) \\
& x_{1t} \le B - S + y_{1t} \quad t \in [1, n], && (11) \\
& x_t \le B z_t \quad t \in [1, n], && (12) \\
& x_t, y_t \ge 0 \quad t \in [1, n], && (13) \\
& z_t \in \{0, 1\} \quad t \in [1, n]. && (14)
\end{aligned}
$$

For the second problem WP2, the feasible set X_2 is given by (10)-(14) plus (7) and (8). Finally, for WP3, the feasible set X_3 is given by (10)-(14) plus (7)-(9).

Our aim is to describe the convex hull of each of the sets X_1, X_2 and X_3 and present tight extended formulations. The main results of the paper are the following theorems.

Theorem 1 *The convex hull of X_1 is given by*

$$y_{1t} \leq S + x_{1,t-1} \qquad t \in [1,n], \tag{15}$$
$$x_{1t} \leq B - S + y_{1t} \qquad t \in [2,n], \tag{16}$$
$$x_t \leq Bz_t \qquad t \in [1,n], \tag{17}$$
$$x_{1t} \leq y_{1t} + \sum_{u\in[1,t]} \min\{x_u, (B-S)z_u\} \qquad t \in [1,n], \tag{18}$$
$$x_t, y_t \geq 0,\ z_t \leq 1 \qquad t \in [1,n]. \tag{19}$$

Theorem 2 *The convex hull of X_2 is given by*

$$y_{1t} \leq S + x_{1,t-1} \quad t \in [2,n], \tag{20}$$
$$x_{1t} \leq (B-S) + y_{1t} \quad t \in [2,n], \tag{21}$$
$$x_t \leq Bz_t \quad t \in [1,n], \tag{22}$$
$$y_t \leq Bw_t \quad t \in [2,n], \tag{23}$$
$$y_1 \leq Sw_1, \tag{24}$$
$$x_{1t} \leq \sum_{u\in[1,t]} \min\{x_u, (B-S)z_u\} + \sum_{u\in[1,t]} \min\{y_u, Sw_u\} \quad t \in [1,n], \tag{25}$$
$$y_{1t} \leq \sum_{u\in[1,t-1]} \min\{x_u, (B-S)z_u\} + \sum_{u\in[1,t]} \min\{y_u, Sw_u\} \quad t \in [2,n], \tag{26}$$
$$x_t, y_t \geq 0,\ z_t, w_t \leq 1 \quad t \in [1,n]. \tag{27}$$

Theorem 3 *The convex hull of X_3 is given by*

$$y_{1t} \leq S + x_{1,t-1} \quad t \in [2,n], \tag{28}$$
$$x_{1t} \leq B - S + y_{1,t-1} \quad t \in [2,n], \tag{29}$$
$$x_1 \leq (B-S)z_1, \tag{30}$$
$$x_t \leq Bz_t \quad t \in [2,n], \tag{31}$$
$$y_1 \leq Sw_1, \tag{32}$$
$$y_t \leq Bw_t \quad t \in [2,n], \tag{33}$$
$$x_{1t} \leq \sum_{u\in[1,t]} \min\{x_u, (B-S)z_u\} + \sum_{u\in[1,t-1]} \min\{y_u, Sw_u\} \quad t \in [2,n], \tag{34}$$
$$y_{1t} \leq \sum_{u\in[1,t-1]} \min\{x_u, (B-S)z_u\} + \sum_{u\in[1,t]} \min\{y_u, Sw_u\} \quad t \in [2,n], \tag{35}$$
$$z_t + w_t \leq 1 \quad t \in [1,n], \tag{36}$$
$$x_t, y_t \geq 0 \quad t \in [1,n]. \tag{37}$$

Introducing the variables π_t and ρ_t for $t \in [1,n]$ with

$$\begin{aligned}
\pi_t &\leq x_t \quad t \in [1,n],\\
\pi_t &\leq (B-S)z_t \quad t \in [1,n],\\
\rho_t &\leq y_t \quad t \in [1,n],\\
\rho_t &\leq Sw_t \quad t \in [1,n],
\end{aligned}$$

we obtain a polynomial size tight extended formulation for X_3 in which the exponential sets of constraints (34) and (35) are replaced by

$$\begin{aligned} x_{1t} &\leq \pi_{1t} + \rho_{1,t-1} \quad t \in [2, n], \\ y_{1t} &\leq \pi_{1,t-1} + \rho_{1t} \quad t \in [2, n]. \end{aligned}$$

The same can be done for X_1 and X_2.

In the remaining part of the paper, we prove these results. In Section 2 we provide properties of extreme points and unit flow formulations based on these properties. In Section 3 we present the proof of the convex hull result for X_3 and discuss briefly how to prove the results for X_1 and X_2. We conclude in Section **??** by showing that the nontrivial inequalities are flow cover inequalities for some single node flow set relaxations.

2. Extreme points and unit flow formulations

To prove the convex hull results, we first characterize the extreme points and then present extended formulations using these characterizations.

Theorem 4 *At an extreme point of* $\text{conv}(X_1)$, $\text{conv}(X_2)$ *and* $\text{conv}(X_3)$, $s_t \in \{0, s_{t-1}, B\}$, $y_t \in \{0, s_{t-1}\}$ *and if* $0 < x_t < B$ *then* $x_t = B - S$, $s_{t-1} = S$, $y_t = 0$ *and* $s_t = B$ *for* $t \in [1, n]$.

Proof. We give the proof for $\text{conv}(X_1)$. The proofs for $\text{conv}(X_2)$ and $\text{conv}(X_3)$ are very similar. Let (x, y, s, z) be in $\text{conv}(X_1)$. Suppose that there exists $t' \in [1, n]$ with $s_{t'} \notin \{0, s_{t'-1}, B\}$ and $s_t \in \{0, s_{t-1}, B\}$ for $t \in [1, t'-1]$. Let t'' be the smallest index in $[t'+1, n]$ with $x_{t''} > 0$ or $y_{t''} > 0$ and let $t'' = n + 1$ if no such index exists. Let ϵ be a very small positive number. Since $s_{t'} \neq s_{t'-1}$, it is not possible to have $x_{t'} = y_{t'} = 0$. In addition, $s_t = s_t'$ and hence $0 < s_t < B$ for $t \in [t'+1, t''-1]$. Also $0 < x_{t''} < B$ if $y_{t''} = 0$.

First look at the case in which $x_{t'} > 0$. We have $x_{t'} < B$ since otherwise $s_{t'} = B$. Consider the solutions (x^1, y^1, s^1, z) and (x^2, y^2, s^2, z) that are the same as (x, y, s, z) except for $x^1_{t'} = x_{t'} - \epsilon$, $x^2_{t'} = x_{t'} + \epsilon$, $s^1_t = s_{t'} - \epsilon$ and $s^2_t = s_{t'} + \epsilon$ for $t \in [t', t''-1]$. If $t'' \neq n+1$, then let $y^1_{t''} = y_{t''} - \epsilon$ and $y^2_{t''} = y_{t''} + \epsilon$ if $y_{t''} > 0$ and $x^1_{t''} = x_{t''} + \epsilon$ and $x^2_{t''} = x_{t''} - \epsilon$ otherwise. As both (x^1, y^1, s^1, z) and (x^2, y^2, s^2, z) are in $\text{conv}(X_1)$ and $(x, y, s, z) = 1/2(x^1, y^1, s^1, z) + 1/2(x^2, y^2, s^2, z)$, (x, y, s, z) is not an extreme point of $\text{conv}(X_1)$.

Now suppose that $x_{t'} = 0$. Then $0 < y_{t'} < B$. In this case, we define the solutions (x^1, y^1, s^1, z) and (x^2, y^2, s^2, z) to be the same as (x, y, s, z) except for $y^1_{t'} = y_{t'} - \epsilon$, $y^2_{t'} = y_{t'} + \epsilon$, $s^1_t = s_{t'} + \epsilon$ and $s^2_t = s_{t'} - \epsilon$ for $t \in [t', t''-1]$. If $t'' \neq n+1$, then let $y^1_{t''} = y_{t''} + \epsilon$ and $y^2_{t''} = y_{t''} - \epsilon$ if $y_{t''} > 0$ and $x^1_{t''} = x_{t''} - \epsilon$ and $x^2_{t''} = x_{t''} + \epsilon$ otherwise. In this case also, one can see that (x^1, y^1, s^1, z) and (x^2, y^2, s^2, z) are in $\text{conv}(X_1)$ and $(x, y, s, z) = 1/2(x^1, y^1, s^1, z) + 1/2(x^2, y^2, s^2, z)$ and as a result (x, y, s, z) is not an extreme point of $\text{conv}(X_1)$. Hence at an extreme point of $\text{conv}(X_1)$, $s_t \in \{0, s_{t-1}, B\}$ for $t \in [1, n]$.

Now let (x, y, s, z) be a an extreme point of $\text{conv}(X_1)$ with $0 < y_{t'} < s_{t'-1}$ for some $t' \in [1, n]$. In this case, $s_{t'}$ cannot be equal to zero. If $s_{t'} = s_{t'-1}$, then $x_{t'} = y_{t'}$ and if $s_{t'} \neq s_{t'-1}$, then $s_{t'} = B$ and $x_{t'} = B - s_{t'-1} + y_{t'}$. In both cases, $0 < x_{t'} < B$. Now, $(x, y, s, z) = 1/2(x^1, y^1, s, z) + 1/2(x^2, y^2, s, z)$, where (x^1, y^1) and (x^1, y^1) are the same as (x, y) except for $x^1_{t'} = x_{t'} - \epsilon$, $x^2_{t'} = x_{t'} + \epsilon$, $y^1_{t'} = y_{t'} + \epsilon$, $y^2_{t'} = y_{t'} - \epsilon$. Since the two points are also in $\text{conv}(X_1)$, (x, y, s, z) cannot be an extreme point.

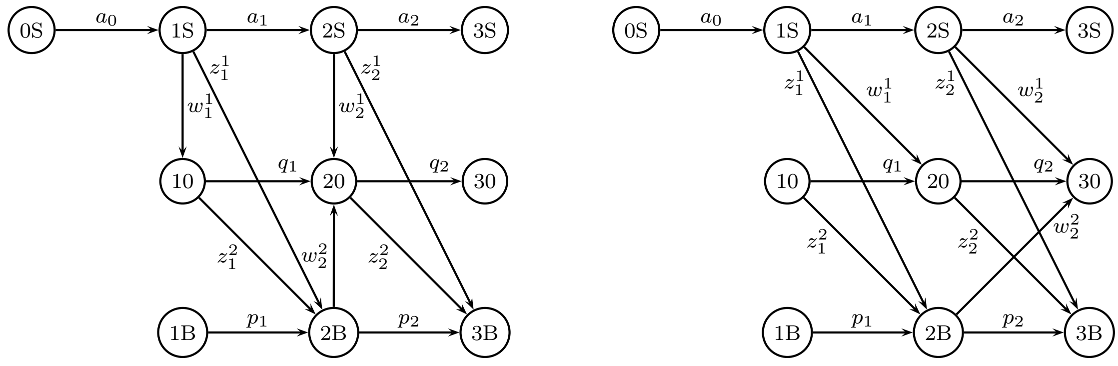

Figure 1: (a) Models 1 and 2 (b) Model 3

Finally, let (x, y, s, z) be a an extreme point of $\text{conv}(X_1)$ with $0 < x_{t'} < B$ for some $t' \in [1, n]$. Since $y_{t'} \in \{0, s_{t'-1}\}$ and $s_{t'-1}, s_{t'} \in \{0, S, B\}$, $x_{t'} = S$ or $x_{t'} = B - S$. Suppose that $x_{t'} = S \neq B - S$. Then $y_{t'} = s_{t'-1}$ and $s_{t'} = S$. Let t'' be the smallest index in $[t'+1, n]$ with $x_{t''} > 0$ or $y_{t''} > 0$ and let $t'' = n + 1$ if no such index exists. Now consider the solutions (x^1, y^1, s^1, z) and (x^2, y^2, s^2, z) that are the same as (x, y, s, z) except for $x^1_{t'} = x_{t'} - \epsilon$, $x^2_{t'} = x_{t'} + \epsilon$, $s^1_t = s_t - \epsilon$ and $s^2_t = s_t + \epsilon$ for $t \in [t', t''-1]$. If $t'' \neq n+1$, then let $y^1_{t''} = y_{t''} - \epsilon$ and $y^2_{t''} = y_{t''} + \epsilon$ if $y_{t''} > 0$ and $x^1_{t''} = x_{t''} + \epsilon$ and $x^2_{t''} = x_{t''} - \epsilon$ otherwise. Since (x^1, y^1, s^1, z) and (x^2, y^2, s^2, z) are in $\text{conv}(X_1)$ and $(x, y, s, z) = 1/2(x^1, y^1, s^1, z) + 1/2(x^2, y^2, s^2, z)$, (x, y, s, z) is not an extreme point of $\text{conv}(X_1)$. Hence $x_{t'} = B - S$, $s_{t'-1} = S$, $y_{t'} = 0$ and $s_{t'} = B$. □

Corollary 5 *At an extreme point of* $\text{conv}(X_1)$, $\text{conv}(X_2)$ *and* $\text{conv}(X_3)$, *there exists* $t' \in [0, n]$ *such that* $s_t = S$ *for* $t \in [1, t']$ *and* $s_t \in \{0, B\}$ *for* $t \in [t'+1, n]$.

Based on the characterization of the extreme points, we provide unit flow formulations for the three warehouse problems. We use the following additional variables in these formulations:

- a_t : 1 if $s_{t-1} = s_t = S$ and $x_t = y_t = 0$ and 0 otherwise,
- p_t : 1 if $s_{t-1} = s_t = B$ and $x_t = y_t = 0$ and 0 otherwise,
- q_t : 1 if $s_{t-1} = s_t = 0$ and $x_t = y_t = 0$ and 0 otherwise,
- w^1_t : 1 if $s_{t-1} = S$ and we sell S units in period t and 0 otherwise,
- w^2_t : 1 if $s_{t-1} = B$ and we sell B units in period t and 0 otherwise,
- z^1_t : 1 if we buy $B - S$ units in period t and 0 otherwise,
- z^2_t : 1 if we buy B units in period t and 0 otherwise.

The unit flow formulations for all three problems are as follows:

- WP1 and WP2:

$$\begin{aligned} a_0 &= 1, \\ a_{t-1} - a_t - z_t^1 - w_t^1 &= 0 \quad t \in [1,n], \\ p_{t-1} + z_{t-1}^1 + z_{t-1}^2 - p_t - w_t^2 &= 0 \quad t \in [2,n], \\ q_{t-1} + w_t^1 + w_t^2 - q_t - z_t^2 &= 0 \quad t \in [1,n], \\ w_1^2 = p_1 = q_0 &= 0, \\ a, z^1, z^2, w^1, w^2, p, q &\geq 0. \end{aligned}$$

- WP3:

$$\begin{aligned} a_0 &= 1, && (38) \\ a_{t-1} - a_t - z_t^1 - w_t^1 &= 0 \quad t \in [1,n], && (39) \\ p_{t-1} + z_{t-1}^1 + z_{t-1}^2 - p_t - w_t^2 &= 0 \quad t \in [2,n], && (40) \\ q_{t-1} + w_{t-1}^1 + w_{t-1}^2 - q_t - z_t^2 &= 0 \quad t \in [2,n], && (41) \\ w_1^2 = z_1^2 = p_1 = q_1 &= 0, && (42) \\ a, z^1, z^2, w^1, w^2, p, q &\geq 0. && (43) \end{aligned}$$

The networks for two periods are depicted in Figure 1.

3. Proof of convex hull results

In the sequel we prove the convex hull result for X_3 in several steps. We first construct an extended formulation for conv(X_3) using the unit flow formulation. Then we project out the additional variables using Fourier-Motzkin elimination in a nontrivial fashion.

Remark 6 *As it models a unit flow problem, the polytope* (38)-(43) *is integral.*

We rewrite the system (38)-(43) in the following equivalent form:

$$\begin{aligned} 1 &= z_{1t}^1 + w_{1t}^1 + a_t \quad t \in [1,n], \\ p_t + w_{2t}^2 &= z_{1,t-1}^1 + z_{1,t-1}^2 + p_1 \quad t \in [2,n], \\ q_t + z_{2t}^2 &= w_{1,t-1}^1 + w_{1,t-1}^2 + q_1 \quad t \in [2,n], \\ w_1^2 = z_1^2 = p_1 = q_1 &= 0, \\ a, z^1, z^2, w^1, w^2, p, q &\geq 0. \end{aligned}$$

Now we eliminate a_t, p_t, q_t by substitution and remove redundancies giving:

$$\begin{aligned} z_{1n}^1 + w_{1n}^1 &\leq 1, && (44) \\ w_{2t}^2 &\leq z_{1,t-1}^1 + z_{1,t-1}^2 \quad t \in [2,n], && (45) \\ z_{2t}^2 &\leq w_{1,t-1}^1 + w_{1,t-1}^2 \quad t \in [2,n], && (46) \\ z^1, z^2, w^1, w^2 &\geq 0, && (47) \\ w_1^2 = z_1^2 &= 0. && (48) \end{aligned}$$

Finally we introduce the additional variables and constraints

$$z_t \geq z_t^1 + z_t^2 \quad t \in [1,n], \tag{49}$$

$$w_t \geq w_t^1 + w_t^2 \quad t \in [1,n], \tag{50}$$

$$x_t = (B-S)z_t^1 + Bz_t^2 \quad t \in [1,n], \tag{51}$$

$$y_t = Sw_t^1 + Bw_t^2 \quad t \in [1,n], \tag{52}$$

$$z_t + w_t \leq 1 \quad t \in [1,n]. \tag{53}$$

Proposition 1 *The polytope (44)-(53) is integral.*

Proof. Suppose that we are minimizing a linear function over the points in this polytope with integer z^1, z^2, w^1, w^2, z and w. Let c_t^z and c_t^w be the objective function coefficients of variables z_t and w_t, respectively, for $t \in [1,n]$. If all these coefficients are zero, since for each solution of (44)-(48), there exist z and w with (49), (50) and (53) and since the polytope defined by (44)-(48) is integral, all optimal solutions lie on a face defined by one of the constraints (44)-(47). Otherwise, let t be such that c_t^z or c_t^w is nonzero. If $c_t^z > 0$, then all optimal solutions satisfy $z_t = z_t^1 + z_t^2$ and similarly if $c_t^w > 0$, then all optimal solutions satisfy $w_t = w_t^1 + w_t^2$. If not, then $c_t^z < 0$ or $c_t^w < 0$ and in this case all optimal solutions lie on the face defined by $z_t + w_t \leq 1$. □

We now introduce two relaxations to help in describing the formulations. Let $k \in [1,n]$. First Q_k

$$x_t \leq Bz_t \; t \in [k+1,n], \tag{54}$$

$$y_t \leq Bw_t \; t \in [k+1,n], \tag{55}$$

$$x_t \geq 0 \; t \in [k+1,n], \tag{56}$$

$$y_t \geq 0 \; t \in [k+1,n], \tag{57}$$

$$x_{1t} \leq (B-S) + y_{1,t-1} \; t \in [k+1,n], \tag{58}$$

$$y_{1t} \leq S + x_{1,t-1} \; t \in [k+1,n], \tag{59}$$

$$z_t + w_t \leq 1 \quad t \in [1,n], \tag{60}$$

and R_k

$$z_1^2 = w_1^2 = 0, \tag{61}$$

$$x_t = (B-S)z_t^1 + Bz_t^2 \; t \in [1,k], \tag{62}$$

$$y_t = Sw_t^1 + Bw_t^2 \; t \in [1,k], \tag{63}$$

$$z_t^1 \geq 0 \; t \in [1,k], \tag{64}$$

$$z_t^2 \geq 0 \; t \in [2,k], \tag{65}$$

$$w_t^1 \geq 0 \; t \in [1,k], \tag{66}$$

$$w_t^2 \geq 0 \; t \in [2,k], \tag{67}$$

$$w_{2t}^2 \leq z_{1,t-1}^1 + z_{2,t-1}^2 \; t \in [2,k], \tag{68}$$

$$z_{2t}^2 \leq w_{1,t-1}^1 + w_{2,t-1}^2 \; t \in [2,k], \tag{69}$$

$$z_t^1 + z_t^2 \leq z_t \; t \in [1,k], \tag{70}$$

$$w_t^1 + w_t^2 \leq w_t \; t \in [1,k]. \tag{71}$$

Theorem 7 *After elimination of $z_t^1, w_t^1, z_t^2, w_t^2$ for $t \in [k+1, n]$, the resulting polyhedron P_k is given by $Q_k \cap R_k$ plus the constraints:*

$$z_{1k}^1 + w_{1k}^1 \leq 1, \quad (72)$$

$$x_{k+1,t} + Sz_{2k}^2 \leq \pi_{k+1,t} + \rho_{k+1,t-1} + S(w_{1k}^1 + w_{2k}^2) \quad t \in [k+1, n], \quad (73)$$

$$y_{k+1,t} + (B-S)w_{2k}^2 \leq \rho_{k+1,t} + \pi_{k+1,t-1} + (B-S)(z_{1k}^1 + z_{2k}^2) \quad t \in [k+1, n]. \quad (74)$$

The proof is by induction over decreasing values of k. We observe that when $k = n$, P_n is the original system (44)-(53). First we consider the case when $k \geq 1$. The case when $k = 0$ will be discussed separately. The passage from P_k to P_{k-1} consists of a series of eliminations, i) elimination of z_k^2 and w_k^2 by substitution, ii) elimination of z_k^1 by Fourier-Motzkin and iii) elimination of w_k^1 by Fourier-Motzkin.

Elimination of z_k^2 and w_k^2

Proposition 2 *Elimination of z_k^2 and w_k^2 by substitution gives $Q_k \cap R_{k-1}$ plus the constraints*

$$\begin{aligned}
Sz_k^1 &\leq Bz_k - x_k, & (75)\\
(B-S)w_k^1 &\leq Bw_k - y_k, & (76)\\
(B-S)z_k^1 &\leq x_k, & (77)\\
Sw_k^1 &\leq y_k, & (78)\\
(B-S)z_k^1 &\geq x_k + B(z_{2,k-1}^2 - w_{1,k-1}^1 - w_{2,k-1}^2), & (79)\\
Sw_k^1 &\geq y_k + B(w_{2,k-1}^2 - z_{1,k-1}^1 - z_{2,k-1}^2), & (80)\\
S(B-S)(z_k^1 + w_k^1) &\geq B(x_{k+1,t} - \pi_{k+1,t} - \rho_{k+1,t-1}) + S(x_k - y_k) & \\
&\quad +BS(z_{2,k-1}^2 - w_{1,k-1}^1 - w_{2,k-1}^2) \quad t \in [k+1, n], & (81)\\
S(B-S)(z_k^1 + w_k^1) &\geq B(y_{k+1,t} - \rho_{k+1,t} - \pi_{k+1,t-1}) + (B-S)(y_k - x_k) & \\
&\quad +B(B-S)(w_{2,k-1}^2 - z_{1,k-1}^1 - z_{2,k-1}^2) \quad t \in [k+1, n], & (82)\\
z_k^1 &\geq 0, & (83)\\
w_k^1 &\geq 0, & (84)\\
z_{1k}^1 + w_{1k}^1 &\leq 1. & (85)
\end{aligned}$$

Proof. We substitute $z_k^2 = (x_k - (B-S)z_k^1)/B$ and $w_k^2 = (y_k - Sw_k^1)/B$. Inequalities (75) and (76) come from (70) and (71), (77) and (78) come from (65) and (67) for $t = k$, (79) and (80) come from (69) and (68) for $t = k$, (81) and (82) come from (73) and (74). (83) and (84) are the inequalities (64) and (66) for $t = k$. □

Elimination of z_k^1

Proposition 3 *Elimination of z_k^1 by Fourier-Motzkin gives $Q_k \cap R_{k-1}$, the constraints (76), (78), (80) and (84) that are unaffected, plus the constraints*

$$x_k \leq Bz_k, \tag{86}$$

$$x_k \geq 0, \tag{87}$$

$$x_k + Sz_{2,k-1}^2 \leq (B-S)z_k + S(w_{1,k-1}^1 + w_{2,k-1}^2), \tag{88}$$

$$z_{1,k-1}^1 + w_{1,k}^1 \leq 1, \tag{89}$$

$$(B-S)w_k^1 \leq (B-S) + y_{1,k-1} - x_{1k}, \tag{90}$$

$$S(B-S)w_k^1 \geq B(x_{kt} - \pi_{kt} - \rho_{k+1,t-1}) + BS(z_{2,k-1}^2 - w_{1,k-1}^1 - w_{2,k-1}^2) - Sy_k \quad t \in [k+1, n], \tag{91}$$

$$S(B-S)w_k^1 \geq B(y_{kt} - \rho_{k+1,t} - \pi_{k,t-1}) + B(B-S)(w_{2,k-1}^2 - z_{1,k-1}^1 - z_{2,k-1}^2) - Sy_k \quad t \in [k+1, n]. \tag{92}$$

Proof. Note that z_k^1 appears in constraints (75), (77), (85) with one sign and constraints (79), (81), (82), (83) with the opposite sign.

We start with inequality (83). Inequalities (86), (87), (89) come from combining (83) with (75), (77), (85) respectively.

Next we use inequalities (81). First note that (75) and (77) can be equivalently written as

$$(B-S)Sz_k^1 \leq \min\{B(B-s)z_k - (B-S)x_k, Sx_k\} = B\pi_k - (B-S)x_k.$$

Combining this with (81) gives

$$\begin{aligned} B\pi_k - (B-S)x_k \;&\geq\; B(x_{k+1,t} - \pi_{k+1,t} - \rho_{k+1,t-1}) + S(x_k - y_k) \\ &+\; BS(z_{2,k-1}^2 - w_{1,k-1}^1 - w_{2,k-1}^2) - S(B-S)w_k^1 \quad t \in [k+1, n], \end{aligned}$$

which is the same as (91).

Combining (81) for $t \in [k+1, n]$ with (85) gives

$$\begin{aligned} S(B-S)(1 - z_{1,k-1}^1 - w_{1,k-1}^1) \geq& B(x_{k+1,t} - \pi_{k+1,t} - \rho_{k+1,t-1}) + S(x_k - y_k) \\ &+ BS(z_{2,k-1}^2 - w_{1,k-1}^1 - w_{2,k-1}^2), \end{aligned}$$

which simplifies to

$$Sx_{1k} + Bx_{k+1,t} \leq S(B-S) + B\pi_{k+1,t} + Sy_{1k} + B\rho_{k+1,t-1},$$

using $x_u = (B-S)z_u^1 + Bz_u^2$ and $y_u = Sw_u^1 + Bw_u^2$ for $u \in [1, k-1]$ and $z_1^2 = w_1^2 = 0$.
We consider an instance when $x_u > (B-S)y_u$ for $u \in T \subseteq [k+1, t]$ and $y_u > Sw_u$ for $u \in V \subseteq [k+1, t-1]$. The inequality takes the form:

$$Sx_{1k} + Bx_T \leq S(B-S) + B(B-S)z_T + Sy_{1k} + By_{[k+1,t-1]\setminus V} + BSw_V.$$

This is dominated by taking $x_{1t} - y_{1,t-1} \leq B - S$ with weight S, $x_u - Bz_u \leq 0$ with weight $B-S$ for $u \in T$, $x_u \geq 0$ for $u \in [k+1, t] \setminus T$ with weight S, $y_u - By_u \leq 0$ with weight S for $u \in V$ and

$y_u \geq 0$ for $u \in [k+1, t-1] \setminus V$ with weight $B - S$, as in

$$\begin{aligned} Sx_{1k} + Sx_{k+1,t} &\leq S(B-S) + Sy_{1k} + Sy_{k+1,t-1}, \\ (B-S)x_T &\leq (B-S)Bz_T, \\ -Sx_{[k+1,t]\setminus T} &\leq 0, \\ 0 &\leq BSw_V - Sy_V, \\ 0 &\leq (B-S)y_{[k+1,t-1]\setminus V}. \end{aligned}$$

Now we use inequalities (82). Combining (75) and (77) in the form $(B-S)Sz_k^1 \leq B\pi_k - (B-S)x_k$ with (82) gives (92).
The inequality obtained from (82) with (85) is

$$(B-S)y_{1k} + By_{k+1,t} \leq S(B-S) + B\rho_{k+1,t} + (B-S)x_{1,k} + B\pi_{k+1,t-1},$$

which is the same as

$$(B-S)y_{1k} + By_T \leq S(B-S) + BSw_T + (B-S)x_{1k} + Bx_{[k+1,t-1]\setminus V} + B(B-S)z_V$$

for $T \subseteq [k+1, t]$ and $V \subseteq [k+1, t-1]$. This is the sum of $B-S$ times $y_{1t} \leq S + x_{1,t-1}$, S times $y_u - By_u \leq 0$ for $u \in T$ and $B-S$ times $y_u \geq 0$ for $u \in [k+1,t] \setminus T$, $B-S$ times $x_u - Bz_u \leq 0$ for $u \in V$ and S times $x_u \geq 0$ for $u \in [k+1, t-1] \setminus V$, and hence is dominated.

Finally we use (79). Inequalities (88) and (90) come from combining (79) with (75) and (85), respectively. Combining (79) with (77) gives $z_{2,k-1}^2 \leq w_{1,k-1}^1 + w_{2,k-1}^2$ that is dominated by (69) for $t = k-1$ as $w_{k-1}^1, w_{k-1}^2 \geq 0$. □

Elimination of w_k^1

Proposition 4 *Elimination of w_k^1 by Fourier-Motzkin gives P_{k-1}.*

Proof. The variable w_k^1 does not appear in constraints (86), (87) and (88). Constraint (88) is the same as (73) for $t = k$. w_k^1 appears in the constraints (80), (84), (91) and (92) with one sign and in (76), (78), (89) and(90) with opposite sign.
Observe that (80) is the same as (92) for $t = k$. We treat it together with (92).

We start with constraint (84). Constraints (55), (57), (72) and (58) for $t = k$ come from combining (84) with (76), (78), (89) and (90), respectively.

Next we combine (92) for $t \in [k, n]$. Combining (76) and (78) with (92) give (74) for $t \in [k, n]$. Inequalities (92) and (89) give

$$(B-S)y_{1k} + B(y_{k+1,t} - \rho_{k+1,t}) \leq S(B-S) + (B-S)x_{1,k-1} + B\pi_{k,t-1} \quad t \in [k, n]. \tag{93}$$

Consider an instance with $y_u > Sw_u$ for $u \in T \subseteq [k+1, t]$ and $x_u > (B-S)z_u$ for $u \in V \subseteq [k, t-1]$. Taking $y_{1t} - x_{1,t-1} \leq S$ with weight $(B-S)$, $y_u \leq Bw_u$ with weight S for $u \in T$, $x_u \leq Bz_u$ with weight $(B-S)$ for $u \in V$, $0 \leq y_u$ for $u \in [k+1, t] \setminus V$ with weight $B-S$ and $0 \leq x_u$ for $u \in [k, t-1]$ with weight S, we see that the inequality is dominated.

Inequalities (92) and (90) give

$$\begin{aligned} &(B-S)y_k + B(y_{k+1,t} - \rho_{k+1,t}) + Sx_{1k} + B(B-S)w_{2,k-1}^2 \\ &\leq S(B-S) + Sy_{1,k-1} + B(B-S)(z_{1,k-1}^1 + z_{2,k-1}^2) + B\pi_{k,t-1}, \end{aligned}$$

that one can rewrite as

$$(B-S)y_{1k} + B(y_{k+1,t} - \rho_{k+1,t}) + Sx_k + BSz^2_{2,k-1} \tag{94}$$
$$\leq S(B-S) + (B-S)x_{1,k-1} + BS(w^1_{1,k-1} + w^2_{2,k-1}) + B\pi_{k,t-1}.$$

Suppose that $y_u > Sw_u$ for $u \in T \subseteq [k+1,t]$ and $x_u > (B-S)z_u$ for $u \in V \subseteq [k,t-1]$.
If $k \in V$, then we take $y_{1,t} - x_{1,t-1} \leq S$ with weight $(B-S)$, $y_u - Bw_u \leq 0$ with weight S for $u \in T$, $x_k + Sz^2_{2,k-1} - (B-S)z_k - S(w^1_{1,k-1} + w^2_{2,k-1}) \leq 0$ with weight B, $x_u - Bz_u \leq 0$ with weight $(B-S)$ for $u \in V \setminus \{k\}$, $0 \leq x_u$ with weight S for $u \in [k+1,t] \setminus V$ and $-y_u \leq 0$ with weight $(B-S)$ for $u \in [k+1,t] \setminus T$. This gives (94).
When $k \notin V$, we replace the third inequality by $z^2_{2,k-1} \leq w^1_{1,k-1} + w^2_{2,k-1}$ with weight BS to show that the inequality is dominated.

Finally we combine inequalities (91) for $t \in [k+1,n]$. (76) and (78) are equivalent to

$$(B-S)Sw^1_k \leq B\rho_k - Sy_k.$$

This combined with (91) give (73) for $t \in [k+1,n]$. The inequalities (91) and (89) give

$$Sx_{1,k-1} + B(x_{kt} - \pi_{kt}) \leq S(B-S) + Sy_{1k} + B\rho_{k+1,t-1}.$$

Inequalities (91) and (90) give

$$Sx_{1,k} + B(x_{kt} - \pi_{kt}) + BSz^2_{2,k-1}$$
$$\leq S(B-S) + Sy_{1k} + BSw^2_{2,k-1} + B\rho_{k+1,t-1}.$$

The proof of dominance for these last two families of inequalities is similar to the cases above with x, y and $S, (B-S)$ interchanged. □

Elimination of z^1_1, w^1_1 from P_1

Finally we consider the elimination of z^1_1, w^1_1 from P_1 as $z^2_1 = w^2_1 = 0$. We obtain $x_1 = (B-S)z^1_1 \leq (B-S)z_1$ and similarly $y_1 \leq Sw_1$. The constraints (73) take the form $x_{1t} - \pi_{1t} \leq \rho_{1,t-1}$ and we note that $\rho_1 = y_1$ as $y_1 \leq Sw_1$. Similarly the constraints (74) take the form $y_{1t} - \rho_{1t} \leq \pi_{1,t-1}$ with $\pi_1 = x_1$ as $x_1 \leq (B-S)z_1$.

This concludes the proof of Theorem 3.

Proofs of Theorems 1 and 2

The proof of Theorem 2 is almost identical to that of Theorem 3. We just note the following changes. We replace (53) with $z_t, w_t \leq 1$ for $t \in [1,n]$. We drop $z^2_1 = 0$ and add $z^2_1 \geq 0$ in every step. Consequently, all the terms of the form sum $z^2_{2\tau}$ are replaced by $z^2_{1\tau}$. In addition, constraint (46) becomes

$$z^2_{1t} \leq w^1_{1t} + w^2_{1t} \quad t \in [1,n],$$

denoted (46^n) .
The changes in the description of P_k are:
(58^n): $x_{1t} \leq (B-S) + y_{1t} \quad t \in [k+1,n]$

(69^n): $z^2_{1t} \le w^1_{1t} + w^2_{2t} \quad t \in [1,k]$
(73^n): $x_{k+1,t} + Sz^2_{1k} \le \pi_{k+1,t} + \rho_{k+1,t} + S(w^1_{1k} + w^2_{2k}) \quad t \in [k+1,n]$

The changes in the system that we obtain after eliminating z^2_k and w^2_k are:
(79^n): $(B-S)(z^1_k + w^1_k) \ge x_k - y_k + B(z^2_{1,k-1} - w^1_{1,k-1} - w^2_{2,k-1})$
(81^n): $S(B-S)(z^1_k + w^1_k) \ge B(x_{k+1,t} - \pi_{k+1,t} - \rho_{k+1,t}) + S(x_k - y_k) + BS(z^2_{1,k-1} - w^1_{1,k-1} - w^2_{2,k-1}) \quad t \in [k+1,n]$

In the elimination of z^1_k, when combining (79^n), the arguments are similar to those above leading to the new version (91^n):

$$S(B-S)w^1_k \ge B(x_{kt} - \pi_{kt} - \rho_{k+1,t}) + BS(z^2_{1,k-1} - w^1_{1,k-1} - w^2_{2,k-1}) - Sy_k.$$

When we combine (79^n) with (75), we obtain (88^n):

$$Bx_k + BSz^2_{1,k-1} \le B(B-S)z_k + BS(w^1_{1,k-1} + w^2_{2,k-1}) + S(y_k + (B-S)w^1_k).$$

Combining (79^n) with (85) gives $x_{1k} \le B - S + y_{1k}$, which is ($58^n$) for $t = k$ and (90) is dropped. Finally, combining (79^n) with (77) gives $Bz^2_{1,k-1} + Sw^1_k \le B(w^1_{1k} + w^2_{2,k-1}) + y_k$ that is a combination of $z^2_{1,k-1} \le w^1_{1,k-1} + w^2_{2,k-1}$, which is ($69^n$) for $t = k-1$, $w^1_k \ge 0$ and $y_k \ge Sw^1_k$.

In the elimination of w^1_k, the variable appears in the constraints (80), (84), (88^n), (91^n) and (92) with one sign and in (76), (78) and (89) with opposite sign.

Combining (84) with (76), (78), and (89) give (55), (57) and (72), respectively.
As there are no changes in (92), combining this inequality with (76) and (78) give (74) for $t \in [k,n]$ and combining them with (89) gives dominated inequalities.

Combining (91^n) for $t \in [k+1,n]$ with (76) and (78) give (73) for $t \in [k+1,n]$. Combining (91^n) with (89) give

$$Sx_{1,k-1} + B(x_{kt} - \pi_{kt}) \le S(B-S) + Sy_{1k} + B\rho_{k+1,t} \tag{95}$$

that are dominated.

Combining (88^n) with (76) and (78) give (73) for $t = k$. Combining (88^n) with (89) gives $S(B-S) + Sy_{1k} + (B-S)Bz_k \ge Sx_{1k} + (B-S)x_k$ which is a combination of $B - S + y_{1k} \ge x_{1k}$ and $Bz_k \ge x_k$.

After eliminating z^1_1 using Proposition 3, we use $w^1_1 = y_1/S$ to obtain the result.

To prove Theorem 1, it suffices to project out the variables w_t from the polyhedron describing $\text{conv}(X_2)$. Constraints (23), (24) disappear, (26) reduces to $0 \le \sum_{u\in[1,t-1]} \min\{x_u, (B-S)z_u\}$ and is dominated and (25) reduces to $x_{1t} \le \sum_{u\in[1,t]} \min\{x_u, (B-S)z_u\} + y_{1t}$.

4. Final Remarks

We terminate with a couple of brief observations relating the model and results to other work. The following result for the constant capacity single node flow set, generalizing Padberg et al. [1985], is well-known and cited in Atamturk et al. [2016], but we have not found an original reference.

Proposition 5 *When b is not a multiple of C, the convex hull of the single node flow set*

$$\sum_{j \in N^+} x_j - \sum_{j \in N^-} x_j \leq b, x_j \leq Cz_j \; j \in N, x \in R_+^{|N|}, z \in \{0,1\}^{|N|},$$

with $N = N^+ \cup N^-, N^+ \cap N^- = \emptyset$ *is obtained by adding the constraints*

$$x_T - (C - \lambda)z_T \leq b - \left\lceil \frac{b}{C} \right\rceil (C - \lambda) + x_{N^- \setminus L} + \lambda z_L$$

where $T \subseteq N^+$*,* $|T| \geq \lceil \frac{b}{C} \rceil$*,* $L \subseteq N^-$ *and* $\lambda = \lceil \frac{b}{C} \rceil C - b$*.*

With $b = B - S$ and $C = B$, we see that inequalities (34) for fixed t are precisely the flow cover inequalities for the single node flow set consisting of (11) for t, (7)-(8) and (12)-(14). Similarly the inequalities (35) are obtained from (10) for fixed t, (7)-(8) and (12)-(14). Thus the convex hull of X_3 is obtained as the intersection of these convex hulls for each fixed t.

It is also natural to view the warehouse model in the lot-sizing context with x_t, z_t as production and set-up variables and y_t, w_t as sales with fixed costs, where there are constant bounds on the stocks, but without fixed demands.

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